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Research Article

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Solution of linear third order ordinary differential equations with variable coefficients by coefficients to be determined

Shengda Tao, Chunyong Wang*

School of Education and Music, Hezhou University, Hezhou 542899, China

Abstract This paper presents the solution formula for the power series solution and the formula for estimating the error of the variable-coefficient linear third-order ordinary differential equation using the undetermined coefficient method. Additionally, we propose a numerical approximation algorithm for the power series solution of the variable-coefficient third-order ordinary differential equation. The algorithm is then verified through an example.

Keywords Third order differential equations with variable coefficients; coefficients to be determined; power series solutions; numerical approximation algorithms; error estimation

1. Introduction

Higher order differential equations with variable coefficients are significant in both theory and practical applications. However, there is no universally applicable analytical method to solve them, including linear second-order ordinary differential equations with variable coefficients. Scholars have attempted to tackle this by employing special transformations to convert specific types of linear third-order ordinary differential equations with variable coefficients into ones with constant coefficients. This allows them to obtain the solutions using established formulas. Literature [1-2] applies the concept of variable transformation to solve a specific type of third-order linear differential equations. In literature [3], the third-order variable coefficient differential equation is transformed into a system of differential equations by replacing the variables. Then, the general solution of the equation is obtained using Liouville's formula and integration. Literature [4] investigates the implementation of Mathematica software for power series solutions and provides a software package for solving power series of general second-order linear differential equations. Literature [5] focuses on the polynomial-type solutions of a specific class of third-order chi-square linear differential equations with polynomial coefficients. It provides the conditions for the existence of polynomial-specific and general solutions of these equations, along with the corresponding algorithms based on the sufficient condition that the chi-square linear equations have non-zero solutions. Literature [6] studies the solution of linear differential equations with variable coefficients, including the exploration of power series solutions for second-degree equations. On the other hand, literature [7] presents the form of special solutions for second-order variable coefficient chi-squared linear differential equations meeting specific conditions. In literature [8], the operator solution of third-order linear equations with variable coefficients is examined using the theory of linear differential operator decomposition, and the method and steps for obtaining such a solution are provided.

This paper is inspired by the literature [6-8] and focuses on finding power series solutions for linear third-order ordinary differential equations with variable coefficients. The method used involves determining the coefficients and providing solution formulae, error estimators, algorithms, and examples to verify the findings.

2. Solution by coefficients to be determined for power series solutions and error estimation

This paper examines differential equations for the following cases

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0 \qquad (1)$$

The initial conditions $y(x_0)$, $y'(x_0)$ and $y''(x_0)$ are known.

Lemma 1[9-10] If $p(x)$, $q(x)$ and $r(x)$ are analytic functions in a neighborhood of a point x_0 at $U(x_0, \delta)$, then they can all be expressed as power series in terms of $x - x_0$. Similarly, the analytic solution $y = y(x)$ of equation (1) can also be expanded as a power series around $x - x_0$ within the domain of x_0 as below.

$$y = \sum_{n=0}^{\infty} d_n (x - x_0)^n. \qquad (2)$$

Equation (1) is solved using the power series solution, which can be expressed by formula (2).The following section derives the power series solution for equation (1) by determining its coefficients.

According to the lemma stated above and Taylor's theorem, the functions $p(x)$, $q(x)$ and $r(x)$ are all able to be expanded as power series in the vicinity of x_0.

$$p(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (3)$$

$$q(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n \qquad (4)$$

$$r(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad (5)$$

where $a_n = \dfrac{p^{(n)}(x_0)}{n!}$, $b_n = \dfrac{q^{(n)}(x_0)}{n!}$ and $c_n = \dfrac{r^{(n)}(x_0)}{n!}$ can be calculated.

By substituting the Taylor expansions (2), (3), (4), and (5) into equation (1), we obtain:

$$\sum_{n=3}^{\infty} n(n-1)(n-2) d_n (x - x_0)^{n-3} + \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{g} \sum_{n=2}^{\infty} n(n-1) d_n (x - x_0)^{n-2} +$$

$$\sum_{n=0}^{\infty} b_n (x - x_0)^n \text{g} \sum_{n=1}^{\infty} n d_n (x - x_0)^{n-1} + \sum_{n=0}^{\infty} c_n (x - x_0)^n \text{g} \sum_{n=0}^{\infty} d_n (x - x_0)^n = 0 \qquad (6)$$

After taking into account the constant terms on each side of Equation (6), we can deduce:

$$3! d_3 + 2! a_0 d_2 + b_0 d_1 + c_0 d_0 = 0 \qquad (7)$$

We can obtain $d_0 = y(x_0)$, $d_1 = \dfrac{y'(x_0)}{1!}$ and $d_2 = \dfrac{y''(x_0)}{2!}$ from the initial conditions $y(x_0)$, $y'(x_0)$ and $y''(x_0)$. According to equation (7), the solution for d_3 can be obtained using the following approach:

$$d_3 = -\frac{1}{3!}\left(2! a_0 d_2 + b_0 d_1 + c_0 d_0\right) \qquad (8)$$

Considering the main terms $x - x_0$ on both sides of equation (6), we get:

$$4 \cdot 3 \cdot 2 \cdot d_4 + (2! a_1 d_2 + 3 \cdot 2 \cdot a_0 d_3) + (2 b_0 d_2 + b_1 d_1) + (c_0 d_1 + c_1 d_0) = 0 \qquad (9)$$

The solution presented in this study is derived from equation (9):

$$d_4 = -\frac{1}{4\cdot 3\cdot 2}\left[(2!a_1 d_2 + 3\cdot 2\cdot a_0 d_3) + (2b_0 d_2 + b_1 d_1) + (c_0 d_1 + c_1 d_0)\right] \qquad (10)$$

In general, by comparing the coefficients of $(x-x_0)^k$ on both sides of equation (6), we can derive the following formula:

$$(k+3)(k+2)(k+1)d_{k+3} + [a_0(k+2)(k+1)d_{k+2} + a_1(k+1)kd_{k+1} + \cdots + a_k 2\cdot 1\cdot d_2] + \left[b_0(k+1)d_{k+1} + b_1 k d_k + \cdots + b_k d_1\right] + \left(c_0 d_k + c_1 d_{k-1} + \cdots + c_k d_0\right) = 0 \qquad (11)$$

The $k+3$ $(k \in N)$ rd coefficient of the power series solution of equation (1) is given as:

$$d_{k+3} = -\frac{\sum_{j=0}^{k} a_j (k+2-j)(k+1-j)d_{k+2-j} + \sum_{j=0}^{k} b_j (k+1-j)d_{k+1-j} + \sum_{j=0}^{k} c_j d_{k-j}}{(k+3)(k+2)(k+1)} \qquad (12)$$

Continuing with the recursion, we can obtain the Taylor coefficients of the unknown function $y = y(x)$ in a neighborhood of x_0. These coefficients are represented as d_0, d_1, d_2 and so on.

According to Taylor's theorem, the residual term for the expansion of equation (1) as a k+3 power series solution is:

$$R_{k+3}(x) = \frac{y^{(k+4)}(\xi)}{(k+4)!}(x-x_0)^{k+4},\ \xi \in \mathrm{U}(x_0)$$

Since $y^{(k+4)}(x)$ is within the resolution $\mathrm{U}(x_0, \delta)$, we can approximate that $y^{(k+4)}(\xi) \approx y^{(k+4)}(x_0)$. Therefore, we obtain:

$$\frac{y^{(k+4)}(\xi)}{(k+4)!} \approx \frac{y^{(k+4)}(x_0)}{(k+4)!} = d_{k+4}$$

The error estimate for the expansion of Equation (1) to the approximate solution of the $k+3$th power series is as follows:

$$|R_{k+3}(x)| \le |d_{k+4}||x-x_0|^{k+4} \le |d_{k+4}|\delta^{k+4} \qquad (13)$$

3. Algorithms for solving power series approximate solutions with error estimates

Based on the derivation process outlined in Section 2, a power series algorithm for approximating the solution of equation (1) is proposed as a solution below:

Step1. The input variable coefficients are denoted as $p(x), q(x)$ and $r(x)$, while the initial conditions are represented by d_0, d_1 and d_2. The expansion point is given as x_0, the desired precision as ε, and the expansion interval as $\mathrm{U}(x_0, \delta)$ The initial expansion order is denoted as k_0.

Setp2. Find the Taylor coefficients of $p(x)$, $q(x)$ and $r(x)$ from Taylor's formula using the formula

$$a_n = \frac{p^{(n)}(x_0)}{n!},\quad b_n = \frac{q^{(n)}(x_0)}{n!},\quad c_n = \frac{r^{(n)}(x_0)}{n!},\quad n = 0,1,2,\cdots$$

Setp3. Use the recursive equation (12) to calculate d_{k+3} for $k = 0,1,\cdots,k_0$.

Step4. Calculate d_{k+4} to obtain the residual estimator:

$$|R_{k+3}(x)| \le |d_{k+4}||x-x_0|^{k+4} \le |d_{k+4}|\delta^{k+4},\quad x \in \mathrm{U}(x_0, \delta).$$

Step5. Calculate the truncation error $\left|R_{k+3}(x)\right|$. If $\left|R_{k+3}(x)\right| \le \varepsilon$, then the solution is complete. Output the power series approximate solution of equation (1), which is $\hat{y} = \sum_{n=0}^{k+3} d_n (x - x_0)^n$ .If $\left|R_{k+3}(x)\right| > \varepsilon$, increment k0 by 1 and proceed to Step 3.

4. Numerical example

Example 1: A linear third order ordinary differential equation with variable coefficients is known

$$y''' + \frac{3}{x}(x^3 - 2)y'' + \frac{3x^6 - 6x^3 + 10}{x^2} y' + x^6 y = 0 \quad (14)$$

The initial conditions are: $y(1) = 3e^{-1/3}$, $y'(1) = 6e^{-1/3}$, $y''(1) = 15e^{-1/3}$.

The analytical solution for equation (14) that satisfies the given initial value condition can be expressed as follows.

$$y = (1 + x^3 + x^6)e^{-x^3/3}.$$

Given the observation interval $x \in [0.4, 1.6]$,the expansion point $x_0 = 1$,the neighborhood radius $\delta = 0.6$, and the allowed error limit $\varepsilon = 0.1$. According to the solution algorithm described in Section 3, when $k = 4$ (i.e., $k + 3 = 7$), the equation (14) is approximated by expanding it into a power series consisting of 7 terms. The resulting approximate solution is given below.

$$\hat{y} = \frac{3}{e^{1/3}} + \frac{6(x-1)}{e^{1/3}} + \frac{15(x-1)^2}{2e^{1/3}} - \frac{75(x-1)^4}{8e^{1/3}} - \frac{153(x-1)^5}{20e^{1/3}} + \frac{211(x-1)^6}{80e^{1/3}} + \frac{187(x-1)^7}{28e^{1/3}}$$

Table 1 below shows the results of the error comparison between the power series approximate solution $\hat{y}$ and the analytical solution $y = y(x)$ in the interval $x \in [0.4, 1.6]$.

Table 1: Results of power series approximate and analytical solution measurements (case k=4)

x	$y = y(x)$	$\hat{y} = \sum_{n=0}^{k+3} d_n (x - x_0)^n$	$\Delta = \lvert y - \hat{y} \rvert$
0.4	1.045551	1.014581	0.03097
0.6	1.17494	1.173819	0.001121
0.8	1.495785	1.495782	3.8E-06
1	2.149594	2.149594	0
1.2	3.212073	3.212071	1.87E-06
1.4	4.516744	4.51659	0.000154
1.6	5.584135	5.589051	0.004916

A graphical comparison is presented below, illustrating the power series approximate solution (denoted as $\hat{y} = \sum_{n=0}^{k+3} d_n (x - x_0)^n$) and the analytical solution (represented by $y = y(x)$).

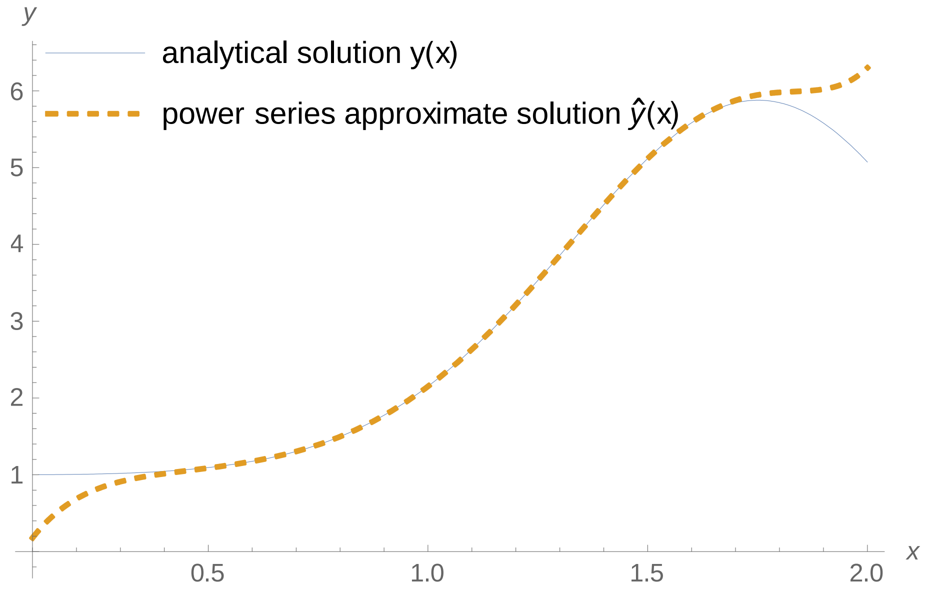

Figure. 1: Graphs of approximate and analytic solutions of power series (case k=4)

The truncation error $\left|R_{k+3}(x)\right|$ of the power series approximation $\hat{y} = \sum_{n=0}^{k+3} d_n \,(x - x_0)^n$ varies and is displayed in the table below.

Table 2: Truncation error of the approximate solution of power series versus order k

k	**0**	**1**	**2**	**3**	**4**	**5**	**6**
$\left\lvert R_{k+3}(x)\right\rvert$	0.870586	0.426239	0.088173	0.133961	0.019436	0.019079	0.007477

The results presented in Table 1 and Fig.1 demonstrate that both the approximate and analytical solutions of the power series expansion of equation (14) up to the seventh power term exhibit significant reduction in errors when compared to the predetermined error limits. Additionally, the function curves of the two solutions show strong agreement. Table 2 demonstrates that truncation errors of the approximate power series solutions are smaller than the given error limits when $k = 4$. Hence, the power series solution effectively approximates the analytic solution when it is expanded to order 7.

5. Conclusion

This paper presents a discussion on the power series solution of linear third-order ordinary differential equations with varying coefficients. The paper provides coefficient solution formulas, error estimators, and algorithms for solving these equations using the determination of coefficients method. Furthermore, the methods and algorithms can be extended to the power series solution of nonlinear third-order or higher-order ordinary differential equations with variable coefficients as well.

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