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Coefficient estimates for certain subclass of analytic and univalent functions associated with the sine hyperbolic function with the complex order

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Abstract In this study, we give some coefficient estimates for the analytic and univalent functions on the open unit disk in the complex plane that are subordinated to sine hyperbolic function with the complex order. For the defined here subclass $S^*_{\sinh}(\tau),\ \tau \in \mathbb{C}-\{0\}$ of analytic and univalent functions with the quantity

$$1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]$$

subordinated to $1+\sinh z$, we obtain coefficient estimates for initial two coefficients and examine the Fekete-Szegö problem for the mentioned class.

Keywords Starlike function, sine hyperbolic function, coefficient estimate, Fekete-Szegö problem, complex order

1. Introduction and preliminaries

In this section, we will give some basic information that we will use in the proof of the main results.

Let $U=\{z\in\mathbb{C}:\ |z|<1\}$ be open unit disk in the complex plane $\mathbb{C}$ and $H(U)$ denote the class of all analytic functions in U. By A, we will denote the class of the functions $f\in H(U)$ given by the following series expansion, explicitly satisfying the conditions $f(0)=0$ and $f'(0)-1=0$

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\cdots+=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}. \tag{1.1}$$

In the literature the class of univalent functions of A is denoted by S. This class first time was introduced by Köebe [1] and has become the core ingredient of advanced research in this field. Bieberbach [2] published a paper in which the famous coefficient hypothesis was proposed. This conjecture states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for all $n\ge 2$. Many researchers worked hard to solve this problem. But for the first time in 1985, it was de-Branges [3], who solved this long-lasting conjecture.

It is well-know that a univalent function $f\in S$ is called a starlike function, if this function maps open unit disk U onto the star-shaped domain of the complex plane. The set of all starlike functions which satisfies the following condition is denoted by S^*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U\ ;$$

that is,

$$S^* = \left\{ f \in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U \right\}.$$

Some of the important and well-investigated subclass of S include the class $S^*(\alpha)$ given below, which called the class of starlike functions of order $\alpha\left(\alpha \in [0,1)\right)$

$$S^*(\alpha) = \left\{ f \in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U \right\}.$$

It is well-known that an analytical function ω satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ is called Schwartz function. As known that two analytic functions f and g which defined in U, it is said that f is subordinate to g and denoted by $f \prec g$, if there exists a Schwartz function ω, such that $f(z) = g(\omega(z))$.

In 1992, Ma and Minda [4] using subordination terminology was defined the class $S^*(\varphi)$ as follows

$$S^*(\varphi) = \left\{ f \in S:\ \frac{zf'(z)}{f(z)} \prec \varphi(z), z \in U \right\},$$

where $\varphi(z)$ is a univalent function with $\varphi(0) = 1$, $\varphi'(0) > 0$ and the region $\varphi(U)$ is star-shaped about the point $\varphi(0) = 1$ and symmetric with respect to real axis. Such a function has a series expansion of the following form

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} b_n z^n,\ b_1 > 0.$$

In the past few years, numerous subclasses of the collection $S^*(\varphi)$ have been introduced as special choices of the function φ. For example, by choosing the function φ as follows

$$\varphi(z) = \frac{1+Az}{1-Bz},\ A \in \mathbb{C}, B \in [-1,0], A \neq B,$$

we obtain the class $S^*(A,B) \equiv S^*\left(\frac{1+Az}{1-Bz}\right)$ which was studied in [5]. For $-1 \le B < A \le 1$, we get the class of Janowski starlike functions investigated in [6] (see also [7]).

Finding bounds for the function coefficients in a given collection is one of the most fundamental problems in geometric function theory.

The list of recent studies on coefficient estimates for the relevant subclasses of the class $S^*(\varphi)$ is given below.

1. $S^*(\varphi,\gamma) = \left\{ f \in S:\ \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, z \in U \right\},\ \gamma \in (0,1]$ (see [8]).
2. $S_L^* = S^*(\varphi)$, with $\varphi(z) = \sqrt{1+z},\ z \in U$ (see [9]).

3. $S^*_{car} = S^*(\varphi)$, with $\varphi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, $z \in U$ (see [10]).
4. $S^*_{\rho} = S^*(\varphi)$, with $\varphi(z) = 1 + \sinh^{-1} z$, $z \in U$ (see [11]).
5. $S^*_{e} = S^*(\varphi)$, with $\varphi(z) = e^z$, $z \in U$ (see [12,13]).
6. $S^*_{\cos} = S^*(\varphi)$, with $\varphi(z) = \cos z$, $z \in U$ (see [14]).
7. $S^*_{\cosh} = S^*(\varphi)$, with $\varphi(z) = \cosh z$, $z \in U$ (see [15]).
8. $S^*_{\tanh} = S^*(\varphi)$, with $\varphi(z) = 1 + \tanh z$, $z \in U$ (see [16]).
9. $S^*_{\sin} = S^*(\varphi)$, with $\varphi(z) = 1 + \sin z$, $z \in U$ (see [17]).
10. $S^*_{\sinh} = S^*(\varphi)$, with $\varphi(z) = 1 + \sinh z$, $z \in U$ (see [18]).

In this paper, we introduce a new subclass of analytic and univalent functions defined on the open unit disk in the complex plane. Here, we give some coefficient estimates for initial two coefficients and examine the Fekete-Szegö problem for the mentioned class.

This study consists of five subtitles. In the first section, we will provide the necessary information about the subject and talk about the studies available in the literature. The second section contains information about the materials and methods adopted to carry out our work. In the third section, we presented the main results obtained in our study. In the fourth section, we summarize the important findings. Finally, in the fifth chapter, the sources we used in our study are given.

2. Materials and methods

In this section, we will describe the new class of analytical and univalent functions, discussing the materials and methods adopted to carry out our work. We will compare the defined class with the classes available in the literature.

As known that the first order of Hankel determinant of the function $f \in S$ is defined as follows

$$H_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

For $\mu \in \mathbb{C}$ or $\mu \in \mathbb{R}$ the functional $H_{2,1}(f, \mu) = a_3 - \mu a_2^2$ is known as the generalized Fekete-Szegö functional [19]. Estimating the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem in the geometric function theory.

Now by using the definition of subordination, we introduce a new subclass of analytic and univalent functions.

Definition 2.1. For $\tau \in \mathbb{C} - \{0\}$ a function $f \in S$ is said to be in the class $S^*_{\sinh}(\tau)$, if the following condition is satisfied

$$1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec 1 + \sinh z, z \in U;$$

that is,

$$S^*_{\sinh}(\tau) \equiv S^*(\tau, 1 + \sinh z) = \left\{ f \in S : \ 1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec 1 + \sinh z \right\}, z \in U.$$

Remark 2.1. In the case $\tau = 1$, we have the class $S^*_{\sinh} \equiv S^*(1 + \sinh z)$ which reviewed in [19].

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\mathrm{Re}(p(z))>0$, $z\in U$, which from the subordination principle easily can written

$$\mathrm{P}=\left\{p\in A:\ p(z)\prec\frac{1+z}{1-z},z\in U\right\},$$

where the function p has the series expansion of the form

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U. \tag{2.2}$$

The class P defined above is known as the class Caratheodory functions in the literature [20].
Now, let's give some lemmas that we will use to prove our main results.
Lemma 2.1 ([19]). Let the function p belong in the class P. Then,

$$|p_n|\le 2 \text{ for each } n\in\mathbb{N} \text{ and } |p_n-\lambda p_k p_{n-k}|\le 2 \text{ for } n,k\in\mathbb{N},\ n>k \text{ and } \lambda\in[0,1].$$

The equalities hold for the function

$$p(z)=\frac{1+z}{1-z}.$$

Lemma 2.2 ([19]) If the an analytic function p given by the form (1.2), then

$$2p_2=p_1^2+(4-p_1^2)x,$$

$$4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)y$$

for $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

3. Main results

In this section, we give upper bound estimates for initial two coefficients and examine the Fekete-Szegö problem for the class $S^*_{\sinh}(\tau)$.

First of all, let's give the following theorem on coefficient estimates.

Theorem 3.1. Let the function $f\in A$ given by (1.1) belong to the class $S^*_{\sinh}(\tau)$, $\tau\in\mathbb{C}-\{0\}$.
Then,

$$|a_2|\le|\tau| \text{ and } |a_3|\le\frac{|\tau|}{2}\begin{cases}1 & \text{if} \quad |\tau|\le 1,\\ |\tau| & \text{if} \quad |\tau|\ge 1.\end{cases}$$

Proof. Let $f\in S^*_{\sinh}(\tau)$, then there exists a Schwartz function ω, such that

$$\frac{zf'(z)}{f(z)}=1+\tau\cdot\sinh\omega(z),\ z\in U.$$

The Caratheodory function $p\in\mathrm{P}$ in terms of Schwartz function ω written as

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+\cdots.$$

It is equivalent to

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots. \tag{3.1}$$

From the series expansion (1.1) of the function $f(z)$, we can write

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \cdots \,. \tag{3.2}$$

Since

$$\sinh z = z + \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots \,, \tag{3.3}$$

from the series expansion (3.1) of the function $\omega(z)$, we have

$$1 + \tau \cdot \sinh \omega(z) = 1 + \frac{\tau}{2} p_1 z + \frac{\tau}{2}\left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots \,. \tag{2.4}$$

Equalizing (3.2) and (3.4), then comparing the coefficients of the same degree terms on the right and left sides, we obtain the following equalities for two initial coefficients of the function $f(z)$

$$a_2 = \frac{\tau}{2} p_1 \,, \tag{3.5}$$

$$a_3 = \frac{\tau(\tau - 1)}{8} p_1^2 + \frac{\tau}{4} p_2 \tag{3.6}$$

Using Lemma 2.1 to the equality (3.5), we can easily see that $|a_2| \le |\tau|$.
By using Lemma 2.2, the equality (3.6) written as follows

$$a_3 = \frac{\tau}{8}\left(\tau p_1^2 + \left(4 - p_1^2\right) x\right), \tag{3.7}$$

where $x \in \mathbb{C}$ with $|x| \le 1$. Applying triangle inequality, from this equality we obtain

$$|a_3| \le \frac{|\tau|}{8}\left(|\tau| t^2 + \left(4 - t^2\right)\xi\right),$$

where $\xi = |x|$ and $t = |p_1|$. If we maximize the function $\varphi : [0,1] \to \mathbb{R}$ defined as

$$\varphi(\xi) = |\tau| t^2 + \left(4 - t^2\right)\xi, \ \xi \in [0,1],$$

we write

$$|a_3| \le \frac{|\tau|}{8}\left(\left(|\tau| - 1\right) t^2 + 4\right), \ t \in [0,2].$$

From this, immediately obtained the desired estimate for $|a_3|$.
Thus, the proof of theorem is completed.
In the case $\tau \in \mathbb{R} - \{0\}$, Theorem 3.1 is given as below.

Theorem 3.2. Let the function $f \in A$ given by (1.1) belong to the class $S^*_{\sinh}(\tau)$, $\tau \in \mathbb{R} - \{0\}$.
Then,

$$|a_2| \le \begin{cases} -\tau & \text{if } \tau < 0, \\ \tau & \text{if } \tau > 0. \end{cases} \quad \text{and} \quad |a_3| \le \begin{cases} \frac{\tau^2}{2} & \text{if } \tau \le -1, \\ \frac{-\tau}{2} & \text{if } -1 \le \tau < 0, \\ \frac{\tau}{2} & \text{if } 0 < \tau \le 1, \\ \frac{\tau^2}{2} & \text{if } \tau \ge 1. \end{cases}$$

Taking $\tau = 1$ in Theorem 3.2, we obtain the following results for $|a_2|$ and $|a_3|$ obtained in [18].

Theorem 3.3. Let the function $f \in A$ given by (1.1) belong to the class S^*_{sinh}. Then,

$$|a_2| \le 1 \text{ and } |a_3| \le \frac{1}{2}.$$

In the following theorem, we give an upper bound estimate for the Fekete-Szegö functional for the class $S^*_{\text{sinh}}(\tau)$.

Theorem 3. 4. Let the function $f \in A$ given by (1.1) belong to the class $S^*_{\text{sinh}}(\tau)$ and $\mu \in \mathbb{C}$. Then,

$$|a_3 - \mu a_2^2| \le \frac{|\tau|}{2} \begin{cases} 1 & \text{if } |1-2\mu||\tau| \le 1, \\ |1-2\mu||\tau| & \text{if } |1-2\mu||\tau| \ge 1. \end{cases}$$

Proof. Let $f \in S^*_{\text{sinh}}(\tau)$ and $\mu \in \mathbb{C}$. Then, from the equalities (2.5) and (2.6), using Lemma 2.2 we can write the following equality for the expression $a_3 - \mu a_2^2$

$$a_3 - \mu a_2^2 = \frac{\tau}{8}\left\{(1-2\mu)\tau p_1^2 + \left(4 - p_1^2\right)x\right\}, \tag{3.8}$$

where $x \in \mathbb{C}$ with $|x| \le 1$. From here, applying triangle inequality we can write

$$|a_3 - \mu a_2^2| \le \frac{|\tau|}{8}\left[|1-2\mu||\tau|t^2 + \left(4-t^2\right)\xi\right], \ \xi \in [0,1], \tag{3.9}$$

with $t = |p_1| \in [0,2]$ and $\xi = |x|$. Maximizing the function

$$\psi(\xi) = |1-2\mu||\tau|t^2 + \left(4-t^2\right)\xi, \ \xi \in [0,1],$$

from the inequality (3.9), we obtain the following inequality

$$|a_3 - \mu a_2^2| \le \frac{|\tau|}{8}\left\{\left[|1-2\mu||\tau|-1\right]t^2 + 4\right\}, \ t \in [0,2]. \tag{3.10}$$

From here obtained the desired result of theorem.

With this, the proof of theorem is completed.

Taking $\tau = 1$ in Theorem 3.4, we obtain the following result for the Fekete-Szegö inequality obtained in [18].

Theorem 3.5. Let the function $f \in A$ given by (1.1) belong to the class S^*_{sinh} and $\mu \in \mathbb{C}$. Then,

$$|a_3 - \mu a_2^2| \le \frac{1}{2} \begin{cases} 1 & \text{if } |1-2\mu| \le 1, \\ |1-2\mu| & \text{if } |1-2\mu| \ge 1. \end{cases}$$

In the case $\mu \in \mathbb{R}$, Theorem 3.4 is given as below.

Theorem 3.6. Let the function $f \in A$ given by (1.1) belong to the class $S^*_{\sinh}(\tau)$ and $\mu \in \mathbb{R}$. Then,

$$\left|a_3-\mu a_2^2\right| \le \frac{|\tau|}{2}\begin{cases} 1 & \text{if} \quad \mu \in \left[\dfrac{|\tau|-1}{2|\tau|}, \dfrac{|\tau|+1}{2|\tau|}\right], \\ |1-2\mu||\tau| & \text{if} \quad \mu \le \dfrac{|\tau|-1}{2|\tau|} \text{ or } \mu \ge \dfrac{|\tau|+1}{2|\tau|}. \end{cases}$$

The proof of this theorem is done similarly to the proof of Theorem 3.4.

Taking $\mu = 0$ and $\mu = 1$ in Theorem 3.6, we get the estimate for $|a_3|$ obtained in Theorem 3.1 and following result for the first order of Hankel determinant.

Corollary 3.1. If the function $f \in A$ given by (1.1) belong to the class $S^*_{\sinh}(\tau)$, then

$$\left|a_3-a_2^2\right| \le \frac{|\tau|}{2}\begin{cases} 1 & \text{if} \quad |\tau| \le 1, \\ |\tau| & \text{if} \quad |\tau| \ge 1. \end{cases}$$

Taking $\tau = 1$ in Corollary 3.1, we obtain the following result obtained in [18].

Corollary 3.2. If the function $f \in A$ given by (1.1) belong to the class $S^*_{\sinh}$, then

$$\left|a_3-a_2^2\right| \le \frac{1}{2}.$$

4. Conclusion

In this study, we gave a coefficient evaluation for the new class $S^*_{\sinh}(\tau)$ of analytical and univalent functions that we defined. We examined both the Fekete-Szegö problem for the defined class and compared the obtained results with those available in the literature. So much so that in the results we found, many results available in the literature are obtained in the specific values of the parameters. We gave these results in the previous section.

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