Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2023, 10(6):131-140**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Coefficient estimates for certain subclass of analytic and univalent functions associated with the sine and cosine functions with the complex order

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Abstract In this paper, we introduce and investigate a new subclass of analytic and univalent functions defined on the open unit disk in the complex plane. We obtain coefficient estimates for initial two coefficients and examine the Fekete-Szegö problem for the defined here subclass, which we will denote $S^*_{\sin,\cos}(\tau)$, $\tau \in \mathbb{C}-\{0\}$ with the quantity

$$1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]$$

subordinated to $\sin z+\cos z$.

Keywords Starlike function, sine function, cosine function, coefficient estimate, Fekete-Szegö problem, complex order

1. Introduction

In this section, we provide some basic information that we will use in the proof of our main results.

By $U=\{z\in\mathbb{C} : |z|<1\}$ and $H(U)$, we will denote open unit disk in the complex plane $\mathbb{C}$ and the class of all analytic functions in U, respectively. Let A be the class of the functions $f\in H(U)$, which satisfied the conditions $f(0)=0$, $f'(0)=1$ and given by

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\cdots+=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}. \qquad (1.1)$$

The class A is known as the class of normalized functions in the literature. The subclass of all univalent functions of A is denoted by S (see [1]). Many mathematicians were interested coefficient estimates problem for this class. Within a short period, Bieberbach [2] published a paper in which the famous coefficient hypothesis was proposed. This hypothesis states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for all $n\ge 2$. Many researchers worked hard to solve this problem. In 1985, it was de-Branges [3], who settled this long-lasting conjecture. There were a lot of study devoted to this conjecture and its related coefficient problems (see [4-17]).

Later, some subclasses of the class S were defined and investigated coefficient estimates problem for these subclasses. It is well-know that a univalent function $f \in S$ is called a starlike function, if this function maps open unit disk U onto the star shaped domain of the complex plane. The set of starlike functions in U satisfied the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U$$

and denoted by S^*; that is,

$$S^* = \left\{ f \in S :\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U \right\}.$$

Some important and well-investigated subclass of S include the class $S^*(\alpha)$ given below, such that is the class of starlike functions of order $\alpha\left(\alpha \in [0,1)\right)$

$$S^*(\alpha) = \left\{ f \in S :\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U \right\}.$$

It is well-known that an analytical function ω satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ is called Schwartz function. For two analytic functions defined in U f and g, we say that f is subordinate to g and denoted by $f \prec g$, if there exists a Schwartz function ω such that $f(z) = g(\omega(z))$.

In 1992, Ma and Minda [4] presented the class $S^*(\varphi)$ using subordination terminology as follows

$$S^*(\varphi) = \left\{ f \in S :\ \frac{zf'(z)}{f(z)} \prec \varphi(z), z \in U \right\},$$

where $\varphi(z)$ is a univalent function with conditions $\varphi(0) = 1$, $\varphi'(0) > 0$ and the region $\varphi(U)$ is star-shaped about the point $\varphi(0) = 1$ and symmetric with respect to real axis. Such a function has the following series expansion

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} b_n z^n,\ b_1 > 0.$$

In the last few years, various subclasses of the class S have been introduced as special choices of the function φ. For example, by choosing the function

$$\varphi(z)=\frac{1+Az}{1-Bz},\ A\in\mathbb{C},B\in[-1,0],A\neq B,$$

we obtain the class $S^*(A,B)\equiv S^*\left(\frac{1+Az}{1-Bz}\right)$, which was studied in [5]. For $-1\le B<A\le 1$, we get the class of Janowski starlike functions investigated in [6] (see also [7]).

The following are recently studied relevant subclasses of the class $S^*(\varphi)$.

1. $S^*(\varphi,\gamma)=\left\{f\in S:\ \frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma},z\in U\right\},\ \gamma\in(0,1]$ (see [8]).
2. $S_L^*=S^*(\varphi)$, with $\varphi(z)=\sqrt{1+z},\ z\in U$ (see [9]).
3. $S_{car}^*=S^*(\varphi)$, with $\varphi(z)=1+\frac{4}{3}z+\frac{2}{3}z^2,\ z\in U$ (see [10]).
4. $S_{\rho}^*=S^*(\varphi)$, with $\varphi(z)=1+\sinh^{-1}z,\ z\in U$ (see [11]).
5. $S_{e}^*=S^*(\varphi)$, with $\varphi(z)=e^z,\ z\in U$ (see [12,13]).
6. $S_{\cos}^*=S^*(\varphi)$, with $\varphi(z)=\cos z,\ z\in U$ (see [14]).
7. $S_{\cosh}^*=S^*(\varphi)$, with $\varphi(z)=\cosh z,\ z\in U$ (see [15]).
8. $S_{\tanh}^*=S^*(\varphi)$, with $\varphi(z)=1+\tanh z,\ z\in U$ (see [16]).
9. $S_{\sin}^*=S^*(\varphi)$, with $\varphi(z)=1+\sin z,\ z\in U$ (see [17]).
10. $S_{\sinh}^*=S^*(\varphi)$, with $\varphi(z)=1+\sinh z,\ z\in U$ (see [18]).

In this paper, we introduce a new subclass of analytic and univalent functions defined on the open unit disk in the complex plane. Here, we give some coefficient estimates for initial two coefficients and examine the Fekete-Szegö problem for the mentioned class.

This study consists of five subtitles. In the first section, we provide the necessary information about the subject and talk about the studies available in the literature. The second section contains information about the materials and methods adopted to carry out our study. In the third section, we presented the main results obtained in our paper. In the fourth section, we summarize the important findings. Finally, in the fifth section, the sources we used in our study are given.

2. Materials and methods

In this section, we describe the new class of analytical and univalent functions, discussing the materials and methods adopted to carry out our study. Here, we compare the defined class with the classes available in the literature.

It is well known that the first order Hankel determinant of the function $f\in S$ is defined as follows

$$H_{2,1}(f)=\begin{vmatrix}1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-a_2^2.$$

For $\mu \in □$ or $\mu \in □$ the functional $H_{2,1}(f,\mu) = a_3 - \mu a_2^2$ is known as the generalized Fekete-Szegö functional [19] in the literature. Estimating the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem in the geometric function theory.

Now using the definition of subordination, we introduce a new subclass of analytic and univalent functions.

Definition 2.1. For $\tau \in □ - \{0\}$ of the function $f \in S$ is said to be in the class $S^*_{\sin,\cos}(\tau)$, if the following condition is satisfied

$$1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \sin z + \cos z, z \in U ;$$

that is,

$$S^*_{\sin,\cos}(\tau) \equiv S^*(\tau, \sin z + \cos z) = \left\{ f \in S : 1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \sin z + \cos z \right\}, z \in U .$$

Remark 2.1. In the case $\tau = 1$, we have class $S^*_{\sin,\cos} \equiv S^*(\sin z + \cos z)$ which reviewed in [20]. Let P be the class of analytic functions in U satisfied the conditions $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$, $z \in U$, such that using the subordination principle, it can be easily written

$$\mathrm{P} = \left\{ p \in A : p(z) \prec \frac{1+z}{1-z}, z \in U \right\},$$

where the function p has the series expansion of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n, \; z \in U . \tag{2.2}$$

The class P defined above is known as the class Caratheodory functions in the literature [21].

Now, let's give some lemmas that we will use to prove our main results.

Lemma 2.1 ([19]). Let the function p belong in the class P. Then,

$$|p_n| \le 2 \text{ for each } n \in □ \text{ and } |p_n - \lambda p_k p_{n-k}| \le 2 \text{ for } n,k \in □,\ n > k \text{ and } \lambda \in [0,1].$$

The equalities hold for the function

$$p(z) = \frac{1+z}{1-z}.$$

Lemma 2.2 ([19]) If the an analytic function p given by the form (1.2), then

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4-p_1^2\right)p_1 x - \left(4-p_1^2\right)p_1 x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right)y$$

for $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

3. Main results

In this section, we give upper bound estimates for initial two coefficients and examine the Fekete-Szegö problem for the class $S^*_{\sin,\cos}(\tau)$.

First of all, let's give the following theorem on coefficient estimates.

Theorem 3.1. Let the function $f \in A$ given by (1.1) belong to the class $S^*_{\sin,\cos}(\tau)$, $\tau \in \mathbb{C} - \{0\}$.

Then,

$$|a_2| \le |\tau| \quad \text{and} \quad |a_3| \le \frac{|\tau|}{2}\begin{cases} 1 & \text{if } |1-2\tau| \le 2, \\ \dfrac{|1-2\tau|}{2} & \text{if } |1-2\tau| \ge 2. \end{cases}$$

Proof. Let $f \in S^*_{\sin,\cos}(\tau)$ and $\tau \in \mathbb{C} - \{0\}$. Then, here exist a Schwartz function $\omega(z)$, such that

$$1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] = \sin\omega(z) + \cos\omega(z).$$

If we write the Caratheodory function $p \in \mathrm{P}$ in terms of Schwartz function ω, we have

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots .$$

It follows from that

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots \tag{3.1}$$

From the series expansion (1.1) of the function $f(z)$, we can write

$$1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right] = 1 + \frac{1}{\tau}\left[a_2 z + \left(2a_3 - a_2^2\right)z^2 + \cdots\right] \tag{3.2}$$

Since

$$\sin z + \cos z = 1 + z - \frac{z^2}{2!} + \cdots , \tag{3.3}$$

using the series expansion (3.1) of the function $\omega(z)$, we have

$$\sin\omega(z)+\cos\omega(z)=1+\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{3}{4}p_1^2\right)z^2+\cdots\ . \tag{3.4}$$

By equalizing (3.2) and (3.4), then comparing the coefficients of the same degree terms on the right and left sides, we obtain the following equalities for two initial coefficients of the function $f(z)$

$$a_2=\frac{\tau}{2}p_1, \tag{3.5}$$

$$a_3=\frac{\tau}{4}\left(p_2+\frac{2\tau-3}{4}p_1^2\right). \tag{3.6}$$

Using Lemma 1.1, from the equality (3.5) we can easily see that $|a_2|\le|\tau|$.

Applying the Lemma 1.2, the equality (3.6) we can write as follows

$$a_3=\frac{\tau}{8}\left(\frac{2\tau-1}{2}p_1^2+\left(4-p_1^2\right)x\right), \tag{3.7}$$

where $x\in\mathbb{C}$ with $|x|\le 1$. Applying triangle inequality, from the equality (3.7) we obtain

$$|a_3|\le\frac{|\tau|}{8}\left(\frac{|2\tau-1|}{2}t^2+\left(4-t^2\right)\xi\right),$$

where $\xi=|x|$ and $t=|p_1|$. If we maximize the function $\varphi:[0,1]\to\mathbb{R}$ defined as follows

$$\varphi(\xi)=\frac{|2\tau-1|}{2}t^2+\left(4-t^2\right)\xi,\ \ \xi\in[0,1],$$

we write

$$|a_3|\le\frac{|\tau|}{8}\left(\frac{|2\tau-1|-2}{2}t^2+4\right),\ t\in[0,2].$$

From this, obtained desired estimate for $|a_3|$.

Thus, the proof of theorem is completed.

In the case $\tau\in\mathbb{C}-\{0\}$, Theorem 2.1 is given as below.

Theorem 3.2. Let the function $f\in A$ given by (1.1) belong to the class $S^{*}_{\sin,\cos}(\tau), \tau\in\mathbb{C}-\{0\}$. Then,

$$|a_2| \le \tau \begin{cases} -1 \text{ if } \tau < 0, \\ 1 \quad \text{if } \tau > 0. \end{cases} \quad \text{and} \quad |a_3| \le \frac{\tau}{2} \begin{cases} \dfrac{2\tau - 1}{2} & \text{if} \quad \tau \le \dfrac{-1}{2}, \\ \text{-1} & \text{if } \tau \in \left[\dfrac{-1}{2}, 0 \right), \\ 1 & \text{if} \quad \tau \in \left(0, \dfrac{3}{2} \right], \\ \dfrac{2\tau - 1}{2} & \text{if} \quad \dfrac{3}{2} \le \tau. \end{cases}$$

Taking $\tau = 1$ in Theorem 3.2, we obtain the following results for $|a_2|$ and $|a_3|$ obtained in [20].

Theorem 3.3. Let the function $f \in A$ given by (1.1) belong to the class $S^*_{\sin,\cos}$. Then,

$$|a_2| \le 1 \quad \text{and} \quad |a_3| \le \frac{1}{2} \ .$$

In the following theorem, we give an upper bound estimate for the Fekete-Szegö functional for the class $S^*_{\sin,\cos}(\tau)$.

Theorem 3. 4. Let the function $f \in A$ given by (1.1) belong to the class $S^*_{\sin,\cos}(\tau)$, $\tau \in \mathbb{R} - \{0\}$ and $\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$. Then,

$$|a_3 - \mu a_2^2| \le \frac{|\tau|}{4} \begin{cases} 2 & \text{if} \quad |2(1-2\mu)\tau - 1| \le 2, \\ |2(1-2\mu)\tau - 1| & \text{if} \quad |2(1-2\mu)\tau - 1| \ge 2. \end{cases}$$

Proof. Let $f \in S^*_{\sin,\cos}(\tau)$ and $\mu \in \mathbb{C}$. From the equalities (3.5) and (3.6), using Lemma 2.2, we can write

$$a_3 - \mu a_2^2 = \frac{\tau}{16} \left\{ \left[2(1-2\mu)\tau - 1 \right] p_1^2 + 2\left(4 - p_1^2\right) x \right\},$$

where $x \in \mathbb{C}$ with $|x| \le 1$. It follows from that

$$|a_3 - \mu a_2^2| \le \frac{|\tau|}{16} \left\{ |2(1-2\mu)\tau - 1| t^2 + 2\left(4 - t^2\right) \xi \right\},$$

with $t = |p_1| \in [0, 2]$ and $\xi = |x|$.

By maximizing the function

$$\psi(\xi) = |2(1-2\mu)\tau - 1| t^2 + 2\left(4 - t^2\right) \xi, \ \xi \in [0,1],$$

we obtain the following inequality

$$\left|a_3-\mu a_2^2\right|\le\frac{|\tau|}{16}\left\{\left[\left|2(1-2\mu)\tau-1\right|-2\right]t^2+8\right\},\ t\in[0,2].$$

From this, we obtain the desired result of theorem.

With this, the proof of the theorem is complete.

In the cases $\tau\in\mathbb{R}-\{0\}$ and $\mu\in\mathbb{R}$, Theorem 3.4 is given as below.

Theorem 3.5. Let the function $f\in A$ given by (1.1) belong to the class $S^*_{\sin,\cos}(\tau)$, $\tau\in\mathbb{R}-\{0\}$ and $\mu\in\mathbb{R}$. Then,

$$\left|a_3-\mu a_2^2\right|\le\frac{|\tau|}{4}\begin{cases}1-2(1-2\mu)\tau & \text{if } \tau<0 \text{ and } \mu\le\frac{1}{2}\left(1+\frac{1}{2\tau}\right),\\ 2 & \text{if } \tau<0 \text{ and } \mu\in\left[\frac{1}{2}\left(1+\frac{1}{2\tau}\right),\frac{1}{2}\left(1-\frac{3}{2\tau}\right)\right],\\ 2(1-2\mu)\tau-1 & \text{if } \tau<0 \text{ and } \mu\ge\frac{1}{2}\left(1-\frac{3}{2\tau}\right),\\ 2(1-2\mu)\tau-1 & \text{if } \tau>0 \text{ and } \mu\le\frac{1}{2}\left(1-\frac{3}{2\tau}\right),\\ 2 & \text{if } \tau>0 \text{ and } \mu\in\left[\frac{1}{2}\left(1-\frac{3}{2\tau}\right),\frac{1}{2}\left(1+\frac{1}{2\tau}\right)\right],\\ 1-2(1-2\mu)\tau & \text{if } \tau>0 \text{ and } \mu\ge\frac{1}{2}\left(1+\frac{1}{2\tau}\right).\end{cases}$$

Taking $\tau=1$ in Theorem 3.4, we obtain the following result for the Fekete-Szegö inequality obtained in [20].

Theorem 3.6. Let the function $f\in A$ given by (1.1) belong to the class $S^*_{\sin,\cos}$ and $\mu\in\mathbb{R}$ or $\mu\in\mathbb{C}$. Then,

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{4}\begin{cases}2 & \text{if } |1-4\mu|\le 2,\\ |1-4\mu| & \text{if } |1-4\mu|\ge 2.\end{cases}$$

Taking $\mu=0$ and $\mu=1$ in Theorem 3.4, we get the second result of Theorem 3.1 and following estimate for the first order Hankel determinant, respectively.

Corollary 3.1. If the function $f\in A$ given by (1.1) belong to the class $S^*_{\sin,\cos}(\tau)$, $\tau\in\mathbb{R}-\{0\}$, then

$$\left|a_3-a_2^2\right|\le\frac{|\tau|}{4}\begin{cases}2 & \text{if } |1+2\tau|\le 2,\\ |1+2\tau| & \text{if } |1+2\tau|\ge 2.\end{cases}$$

In the case $\tau \in \mathbb{R} - \{0\}$, from the Corollary 3.1 we obtain the following result.

Corollary 3.2. If the function $f \in A$ given by (1.1) belong to the class $S^{*}_{\sin,\cos}(\tau), \tau \in \mathbb{R} - \{0\}$, then

$$\left|a_3 - a_2^2\right| \le \frac{\tau}{4}\begin{cases} 2\tau+1 & \text{if } \quad \tau \le -\dfrac{3}{2}, \\ -2 & \text{if } \quad -\dfrac{3}{2} \le \tau < 0, \\ 2 & \text{if } \quad 0 < \tau \le \dfrac{1}{2}, \\ 2\tau+1 & \text{if } \quad \dfrac{1}{2} \le \tau. \end{cases}$$

Taking $\tau = 1$ in Corollary 3.2, we obtain the following result for the first order Hankel deterinant obtained in [20].

Corollary 3.3. If the function $f \in A$ given by (1.1) belong to the class $S^{*}_{\sin,\cos}$, then

$$\left|a_3 - a_2^2\right| \le \frac{3}{4}.$$

4. Conclusion

In this study, we gave a coefficient estimates for defined first time new subclass of analytical and univalent functions. We examine both the Fekete-Szegö problem for the defined class and compared the obtained results with those available in the literature. So much so that from the results we found, many results available in the literature are obtained in the specific values of the parameters. We gave these results in the previous section.

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