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Research Article

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Numerical Study of Navier-Stokes Equations in Vorticity/Stream Function Formulation on the Torus in Two-Dimensional Flow under MATLAB

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Abstract The Navier-Stokes equations can be rearranged in 2D incompressible flows in terms of stream function and vorticity. In many areas of applications, the stream function and vorticity form of the Navier Stokes equations shows important effect into the mechanisms leading the flow than the first variables formulation. It's also useful for numerical study it avoids some problems resulting from the discretization. In this work a numerical approach with the stream function and vorticity is used with a discretization of the governing equations and the boundary conditions in the domain included in a square of unit length where the upper boundary (the lid) at y = 1, moves with a constant velocity U = 1. The Reynolds number based on the size of the domain, the velocity of the moving wall, density ρ = 1 and viscosity μ = 0.001 is Re = 500. A code is developed under MATLAB software. The evolution of the vorticity field is determined in term of time. To do this, a differential finite method is used.

Keywords Navier, Stockes, stream function, vorticity, Flow, MATLAB

Introduction

Most of the fluid flows [1] with which we are familiar, from bathtubs to swimming pools, are not rotating, or they are rotating so slowly that rotation is not important except maybe at the drain of a bathtub as water is let out. As a result, we do not have a good intuitive understanding of rotating flows [2]. In simple words, vorticity [3] is the rotation of the fluid. The rate of rotation can be defined various ways. Consider a bowl of water sitting on a table in a laboratory. The water may be spinning in the bowl. In addition to the spinning of the water, the bowl and the laboratory are rotating because they are on a rotating earth. The two processes are separate and lead to two types of vorticities.

The stream function-vorticity formulation was among the first unsteady, incompressible Navier– Stokes [4] algorithms. The original finite difference [5] algorithm was developed by Fromm [6] at LosAlamos laboratory. For incompressible two-dimensional flows [7] with constant fluid properties, the Navier–Stokes equations can be simplified by introducing the stream function ψ [8] and vorticity ω as dependent variables. T Kozlowski and Al show the generation of vorticity field by flapping the profile [9], M. Forman and Al [10] studied the Vorticity Confinement method applied to flow around an Ahmed body and comparison with experiments. A numerical study is done in this work in two-dimensional flow Navier-stokes equations in vorticity/stream function formulation on the torus.

Materials and Methods

The vorticity vector at a point is defined as twice the angular velocity and is

$$\bar{\omega} = \nabla \times \bar{\nabla} \quad (1)$$

which, for two-dimensional flow in x-y plane, is reduced to

$$\omega_z = \bar{\omega}.\mathrm{k} = \frac{\partial \mathrm{v}}{\partial \mathrm{x}} - \frac{\partial u}{\partial y} \quad (2)$$

For two-dimensional, incompressible flows, a scalar function may be defined in such a way that the continuity equation [11] is identically satisfied if the velocity components, expressed in terms of such a function, are substituted in the continuity equation.

The stream function is given by $\bar{\omega} = \nabla \times \psi . \bar{k}$ (3)

In Cartesian coordinate system, the above relation becomes.

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \quad (4)$$

Lines of constant ψ are streamlines (lines which are everywhere parallel to the flow), giving this variable its name.

Now, a Poisson equation for ψ can be obtained by substituting the velocity components, in terms of stream function, in the equation (2). Thus, we have

$$\nabla^2 \psi = -\omega \quad (5)$$

where the subscript z is dropped from ωz. This is a kinematic equation connecting the stream function and the vorticity. So if we can find an equation for ω we will have obtained a formulation that automatically produces divergence-free velocity fields.

Finally, by taking the curl of the general Navier–Stokes equation, we obtain the following Helmholtz equation:

$$\frac{\partial \bar{\omega}}{\partial x} + \bar{\nabla} \times \nabla \bar{\omega} = \nabla \times \bar{f_e} + \bar{\omega} . \nabla . \bar{V} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nabla \times (\frac{1}{\rho} \nabla . \bar{\bar{\tau}}) \quad (6)$$

For a Newtonian fluid [12] with constant kinematic viscosity coefficient ν, the viscous stress term reduces to the Laplacian of the vorticity

$$\nabla \times \left(\frac{1}{\rho} \nabla . \bar{\bar{\tau}}\right) = v . \nabla^2 \omega \quad (7)$$

For incompressible flow with constant density, the third and fourth terms on the right-hand side become zero. Further, in the absence of body force, Helmholtz equation [13] reduces to

$$\frac{\partial \bar{\omega}}{\partial x} + \bar{\nabla} \times \nabla \bar{\omega} = \bar{\omega} . \nabla . \bar{V} + v . \nabla^2 \omega \quad (8)$$

For two-dimensional flows, the term, ω ·∇V = 0 by continuity equation (3), and the Helmholtz equation further reduces to a form:

$$\frac{\partial \bar{\omega}}{\partial x} + \bar{\nabla} \times \nabla \bar{\omega} = v . \nabla^2 \omega \quad (9)$$

This parabolic PDE is called the vorticity transport equation.

The substitution of vorticity defined by (2) in the two-dimensional Navier–Stokes equation for incompressible flow [14] without body force term gives the vorticity transport equation from the scalar form of momentum:

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = v \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \quad (10)$$

The relation connecting the stream function and vorticity (6) is listed below:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \quad (11)$$

Equations (14) and (15) form the system PDEs for stream function-vorticity formulation

Boundary conditions

We will consider this in the context of a classical problem which has wall boundaries surrounding the entire computational region, the so-called 'lid-driven cavity' problem depicted in figure 1. We consider the domain included in a square of unit length, with 0 ≤ x, y ≤ 1, where the upper boundary (the lid) at y = 1, moves with a constant velocity U = 1. The Reynolds number based on the size of the domain, the velocity of the moving wall, density ρ = 1 and viscosity μ = 0.001 is Re = 500.

The boundary conditions are shown in the figure below.

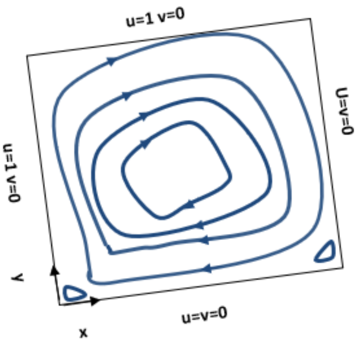


Figure 1 Cavity flow-Stream line

The conditions are:

Since flow is parallel to the walls of the cavity, walls may be treated as streamline. Thus, the stream function value on the wall streamline is set as a constant. That is ψ = c, where c is an arbitrary constant, which may be set equal to zero. Let us examine the application of boundary conditions on a solid wall. Since stream function is a constant along a wall, all the derivatives of stream function along the wall vanish. Hence, the Poisson equation for stream function (11) reduces to

$$u(x,0)=0 \quad u(x,1)=0 \quad u(0,y)=0 \quad u(1,y)=0 \quad v(x,0)=0 \quad v(x,1)=0 \quad v(0,y)=0 \quad v(1,y)=0$$

2.2. Discretisation of equations

Essentially, the system is composed of the vorticity transport equation (9) and the Poisson equation for stream function (11).

Here we use the explicit FTCS scheme, where Euler forward difference for temporal derivative and central differences for space derivatives are used.

$$\frac{\omega_{ij}^{n+1}-\omega_{ij}^{n}}{\Delta t}+u_{ij}\left(\frac{\omega_{i+1,j}^{n+1}-\omega_{i-1,}^{n}}{2\Delta x}\right)+v_{ij}\left(\frac{\omega_{i+1,j}^{n+1}-\omega_{i,j-1}^{n}}{2\Delta y}\right)=\nu\left(\frac{\omega_{i+1,j}^{n}-2\omega_{ij}^{n}+\omega_{i-1,j}^{n}}{2\Delta x^2}+\frac{\omega_{i,j+1}^{n}-2\omega_{ij}^{n}+\omega_{i,j-1}^{n}}{2\Delta y^2}\right) \quad (12)$$

The stability conditions for the FTCS scheme are.

$$d=\frac{\nu\Delta t}{\Delta x^2}+\frac{\nu\Delta t}{\Delta y^2}\leq\frac{1}{2} \qquad \text{or} \qquad R_{e\Delta x c_x}+R_{e\Delta y c_y}\leq 2$$

Where

$$R_{e\Delta x}=\frac{u\Delta x}{V} \qquad R_{e\Delta y}=\frac{V\Delta y}{V} \qquad c_x=\frac{u\Delta t}{\Delta x} \qquad c_y=\frac{u\Delta t}{\Delta x}$$

Therefore, the use of upwind type differencing scheme may be more appropriate if the flow field is convection dominated. However, first-order upwind scheme is too diffusive and may not be suitable for practical applications. Thus, we use the second-order upwind scheme for the discretization of convection terms. The discretized vorticity transport equation can be Witten as

$$\frac{\omega_{ij}^{n+1}-\omega_{ij}^{n}}{\Delta t}+u_{ij}\left(\frac{\omega_{i+1,j}^{n+1}-\omega_{i-1,}^{n}}{2\Delta x}\right)+q(u^{+}\omega_x^{-}+u^{-}\omega_x^{+})+v_{ij}\left(\frac{\omega_{i+1,j}^{n+1}-\omega_{i,j-1}^{n}}{2\Delta y}\right)+q\left(v^{+}\omega_y^{-}+v^{-}\omega_y^{+}\right)=$$
$$\nu\left(\frac{\omega_{i+1,j}^{n}-2\omega_{ij}^{n}+\omega_{i-1,j}^{n}}{2\Delta x^2}+\frac{\omega_{i,j+1}^{n}-2\omega_{ij}^{n}+\omega_{i,j-1}^{n}}{2\Delta y^2}\right) \quad (13)$$

$u^{-}=\min(u_{ij}^{n},0) \quad u^{+}=\max(u_{ij}^{n},0)$

$v^{-}=\min(v_{ij}^{n},0) \quad v^{+}=\max(v_{ij}^{n},0)$

$$\omega_x^- = \frac{\omega_{i-2,j}^n - 3\omega_{i-1,j}^n + 3\omega_{ij}^n - \omega_{i+1,j}^n}{3\Delta x}$$

$$\omega_x^+ = \frac{\omega_{i-1,j}^n - 3\omega_{ij}^n + 3\omega_{i+1j}^n - \omega_{i+2,j}^n}{3\Delta x}$$

$$\omega_y^- = \frac{\omega_{i,j-2}^n - 3\omega_{i,j-2}^n + 3\omega_{ij}^n - \omega_{i,j+1}^n}{3\Delta x}$$

$$\omega_y^+ = \frac{\omega_{i,j-1}^n - 3\omega_{ij}^n + 3\omega_{ij+1}^n - \omega_{i,j+2}^n}{3\Delta x}$$

It may be noted that q = 0.5 represents the third-order accurate upwind formula and for other values of q, the modified formula is only second-order accurate. Also, q = 0 corresponds to the central difference scheme.

The stream function equation (11) is solved at every time step using an appropriate numerical scheme. Since it is an elliptic equation, we use the standard central differencing scheme for discretization of second order spatial derivatives. The discretized equation is given as follows.

$$\frac{\psi_{i+1,j}^{n+1} - 2\psi_{ij}^{n+1} + \psi_{i-1,j}^{n+1}}{\Delta x^2} + \frac{\psi_{i,j+1}^{n+1} - 2\psi_{ij}^{n+1} + \psi_{i,j-1}^{n+1}}{\Delta y^2} = -\omega_{ij}^{n+1} \quad (14)$$

A similar procedure is used to derive the boundary conditions at right, bottom, and top wall. The appropriate expressions are.

$$\omega_{M,j} = -\frac{\partial^2\psi}{\partial x^2}|M,j = \frac{2(\psi_{M,j} + \psi_{M-1,j})}{\Delta x^2} + \frac{2v_{M,j}}{\Delta x} \quad (15)$$

$$\omega_{i,1} = -\frac{\partial^2\psi}{\partial y^2}|i,1 = \frac{2(\psi_{i,1} + \psi_{i,2})}{\Delta y^2} + \frac{2u_{i,1}}{\Delta y}$$

$$\omega_{i,N} = -\frac{\partial^2\psi}{\partial y^2}|i,N = \frac{2(\psi_{i,N} + \psi_{i,N})}{\Delta y^2} - \frac{2u_{i,N}}{\Delta y} \quad (16)$$

Results & Discussion

A solution algorithm for computing evolution of incompressible, two-dimensional flow using stream function vorticity formulation under MATLAB is given as follows:

1. Initialize the velocity field and compute the associated vorticity field and stream function field using equations (2) and (11).
2. Compute the boundary conditions for vorticity.
3. Solve the vorticity transport equation (10) to compute the vorticity at new time step; any standard time marching scheme may be used for this purpose.
4. Solve the Poisson equation [15] for stream function (15) to compute the stream function field at new time step; any iterative scheme for elliptic equations may be used.
5. Compute the velocity field at new time step using the relations (5).
6. Return to step 2 and repeat the computation for another time step.

The algorithm is computed under MATLAB software by using the following values.

viscosity : u=1.0e-3;

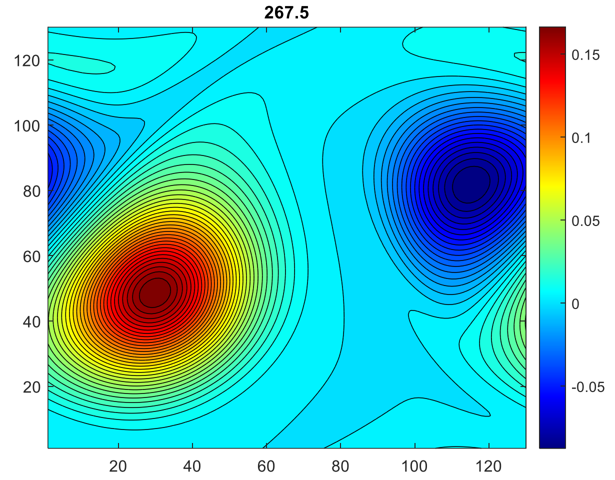
resolution suivant x NX=130;

% resolution suivant y NY=130;

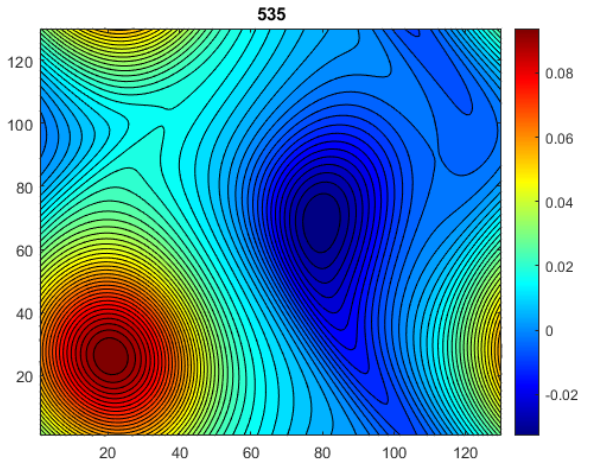
Pas de temps dt=1e-1;

Temps final TF=2000.0;

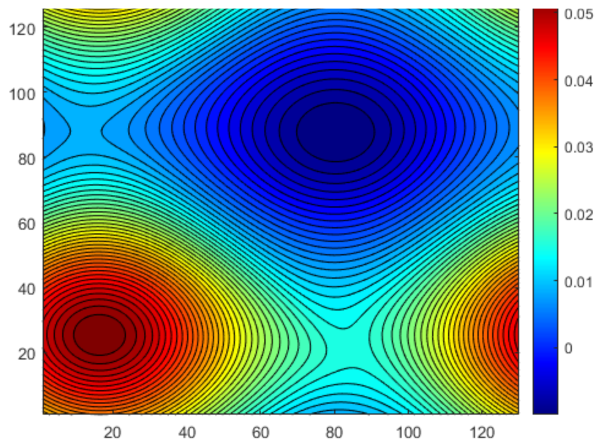
The results are shown in figures below:



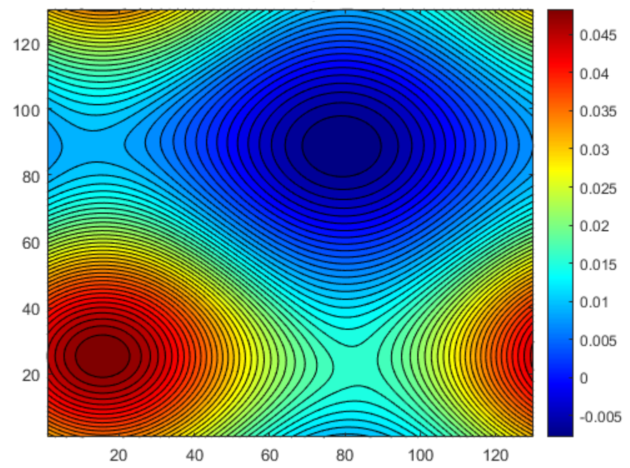
a) Evolution of the vorticity field and velocity after 267.5s



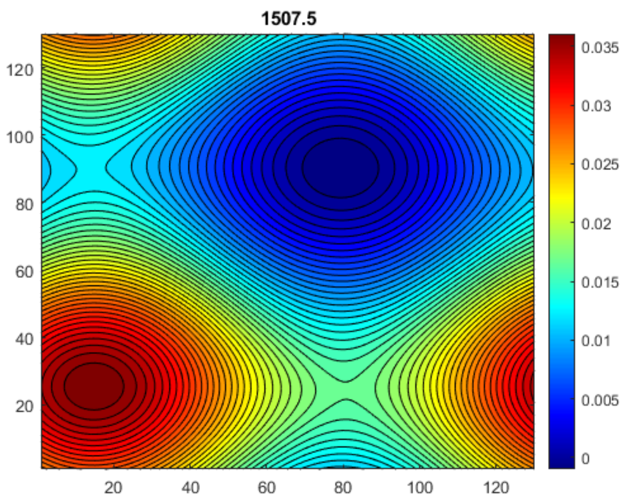
b) Evolution of the vorticity field and velocity after 535s



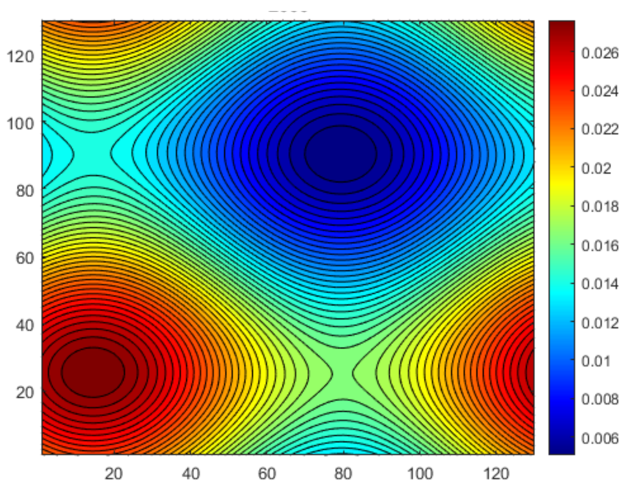
c) Evolution of the vorticity field and velocity after 1015s seconds



d) Evolution of the vorticity field and velocity after 1112.5s seconds



e) Evolution of the vorticity field and velocity after 1507.5s



f) Evolution of the vorticity field and velocity after 2000s

These figures above show the evolution of the vorticity field and the velocity in a square domain at specified time. The velocity is decreasing over the time 2000 seconds. The magnitude of the velocity is important at the wall of the square domain and that is visible on each figure compared to the center of the domain. Also, the surface covers by weak velocity flow are greater than the surface cover by the high velocity of flow. The rotational flow appears clearly after 500s.The results the algorithm method applied under MATLAB is stable. Nevertheless, this code doesn't not work for viscosity smaller than 1.0 10^{-3} in a square domain. At the total time 2000 seconds the vorticity field and velocity appear clear on the square domain.

4. Conclusion

The vorticity-stream function approach has seen considerable use for two-dimensional incompressible flows. It has become less popular in recent years because its extension to three-dimensional flows is difficult. Both the vorticity and stream function become three-component vectors in three dimensions so one has a system of six partial differential equations in place of the four that are necessary in a velocity-pressure formulation. It also inherits the difficulties in dealing with variable fluid properties, compressibility, and boundary conditions that were described above for two dimensional flows.

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