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Numerical Simulation of Shallow Water Equations with Moving Grid Method

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Abstract The moving grid method is used to determine the numerical solution of shallow water equations. To illustrate the efficiency and accuracy, we compare the computed solutions with a reference one obtained using a very fine mesh on two test problems.

Keywords moving grid, monitor function, arc-length monitor.

1. Introduction

Many problems involving shallow water flow in oceanography and atmospheric sciences are modeled by shallow water equation system. It's a hyperbolic system of conservation laws that describes various flows such as rivers, coastal areas, dam breaks, flooding, flow of pollutants, tsunamis, avalanches [1, 2]. The shallow water equations with a non-flat bottom topography play a critical role in the modeling and simulation of the flows in rivers, channels and coastal areas. These equations are a nonlinear system of partial differential equations, i.e., conservation laws describing the evolution of the height and mean velocity of the fluid. In general, solutions of these equations are not available, except for certain with simplified conditions. Due to the nonlinearity of the model, the complexity to the applications, much numerical methods have been developed to solve these equations approximately. A wide range of numerical schemes based on the finite difference, finite element and finite volume methods have been applied to solve numerically the solution of these equations. However, many real applications introduce complications, the main problem in solving the shallow water equations is the presence of the source terms modeling the bottom topography and the Coriolis forces included in the system so, it is very important to have an accurate, efficient and robust numerical method for the shallow water equation system. Some numerical techniques to solve time dependent partial differential equations (PDEs) integrate on a uniform spatial grid that is kept fixed on the entire time interval and when the solutions have regions of high spatial activity, a standard fixed grid technique is inefficient, so, to achieve an accurate numerical approximation, we use a very large number of grid points. The grid on which the PDE is discretized then needs to be locally refined. Moreover, if the regions of high spatial activity are moving in time, then techniques are needed to also adapt the grid in time [3]. The aim is to use the technique of moving grid method to solve shallow water equations under the method of lines. This paper is organized as follows: In Section 2, we give a brief review of the method of lines and the moving grid method. In Section 3, we apply the method to the shallow water system. Some numerical results are shown in Section 4. Concluding remarks are given in Section 5.

2. Brief review of the method of lines and the moving grid method

2.1 Method of lines

Method of lines is a semi-discrete approach that involves reducing an initial/boundary value problem to a system of ordinary differential equations in time using discretization in space. The most important advantage of the method approach is that it is possible to achieve higher-order approximations in the discretization of spatial derivatives without significant increasing in the computational complexity. The accuracy of the method can be enhanced using a highly reliable and robust ODE solvers. The method is stable and suitable even for strong shock waves problems. To apply the method of lines, we must: partition the solution region into layers, discretize the partial differential equation in a coordinate direction, transform it to obtain decoupled ordinary differential equations, reverse transform and introduce boundary conditions, then resolve the resulting system [4, 5, 6].

Let us consider, in 1D, the following general problem:

$$\begin{cases} u_t = \mathcal{L}(u, x, t), \qquad a < x < b, t > 0 \\ u(x, 0) = u^0(x), \ \ (IC) \qquad (1) \\ \mathcal{B}(u, x, t) = 0, x = a; x = b, (BCs) \end{cases}$$

where $\mathcal{L}, \mathcal{B}$ are a given differential operator, the order of $\mathcal{B}$ is less than the order of $\mathcal{L}$.

According to the method of lines, the coordinate x is discretized with n uniformly spaced grid points x_i. The partial derivatives depending on spatial variable in system (1) are replaced by, for example finite difference method approximations, finite volume method approximations, at grid point x_i and this yields a system of ordinary differential equations which depend on t in the following form:

$$\frac{du_i}{dt} = f(u_i), i = 1,2, \dots, n. \quad (2)$$

This system (2) can be solved by using ODE solver like ode15s of MATLAB.

2.2 The governing shallow water equation

The system of the shallow water equations in one dimension under specific assumptions are as follows [1, 7, 8]:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \qquad (3) \\ \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) + gh\frac{\partial B}{\partial x} = 0, \qquad (4) \end{cases}$$

where $h(x, t)$ is the water depth, $u(x, t)$ is the velocity, $B(x, t)$ is the bottom elevation and g is the gravitational constant. Sometime one can use the notation $q(x, t) = h(x, t)u(x, t)$ which is the discharge and then, the system $(3) - (4)$ become:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \qquad (5) \\ \frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{1}{2}gh^2\right) + gh\frac{\partial B}{\partial x} = 0, (6) \end{cases}$$

2.3 The numerical scheme of moving grid method

Suppose that $[a;\ b]$ is the physical domain with a physical variable x and $[0;\ 1]$ is the computational domain for a computational variable ξ. The coordinates transform is expressed as follow [9, 10]:

$$x = x(\xi;\ t): [0, 1] \longrightarrow [a, b], t > 0, x \in [a; b], \xi \in [0; 1]$$

Thus, the solution h, u are transformed as:

$$h(x;\ t) = h(x(\xi, t);\ t), (7)$$
$$u(x;\ t) = u(x(\xi, t);\ t), (8)$$

The coordinate x is rearranged as follows:

$$x_i(\xi) = x(\xi_i, t), i = 1, n + 1.$$

$$\xi_i = \frac{(i-1)(b-a)}{n}, i = 1, n+1.$$

The uniform mesh on $[0, 1]$ is ξ_i and

$$a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$$

is the corresponding mesh on physical domain. Applying the chain rule of the method

$$h_x = \frac{h_\xi}{x_\xi}, \quad h_t = \dot{h} - \frac{h_\xi}{x_\xi} x_t$$

$$q_x = \frac{q_\xi}{x_\xi}, \quad q_t = \dot{q} - \frac{q_\xi}{x_\xi} x_t$$

$$B_x = \frac{B_\xi}{x_\xi}$$

Using the method of lines method and the centered finite difference scheme, we obtain the following ODEs system:

$$\begin{cases} \dfrac{dh_i}{dt} - \dfrac{h_{i+1} - h_{i-1}}{x_{i+1} - x_{i-1}} \dfrac{dx_i}{dt} + \dfrac{q_{i+1} - q_{i-1}}{x_{i+1} - x_{i-1}} = 0; \qquad i = 2, \dots, n, (10) \\ \dfrac{dq_i}{dt} - \dfrac{q_{i+1} - q_{i-1}}{x_{i+1} - x_{i-1}} \dfrac{dx_i}{dt} + \dfrac{2q_i}{h_i} \dfrac{q_{i+1} - q_{i-1}}{x_{i+1} - x_{i-1}} - \left(\dfrac{q_i}{h_i}\right)^2 \dfrac{h_{i+1} - h_{i-1}}{x_{i+1} - x_{i-1}} + gh_i \left(\dfrac{h_{i+1} - h_{i-1}}{x_{i+1} - x_{i-1}} - \dfrac{B_{i+1} - B_{i-1}}{x_{i+1} - x_{i-1}}\right) = 0, (11) \end{cases}$$

In the moving grids method, the monitor function connecting the mesh with the physical solution, is chosen to redistribute more grid points at critical regions where more accuracy is needed there by reducing errors introduced by the numerical scheme [11, 12]. In this paper, arc-length monitor function is used with MATLAB solver ode15s for the numerical simulation. We give a summary of the calculation statistics using the following notations:

$\boldsymbol{n}$: moving grid node number,
$\boldsymbol{nr}$: grid fixe node number,
$\boldsymbol{STEPS}$: number of successful steps,
$\boldsymbol{FAIL}$: number of failed attempts,
$\boldsymbol{FNS}$: number of function evaluations,
$\boldsymbol{PDR}$: number of partial derivatives,
$\boldsymbol{LU}$: number of LU decompositions,
$\boldsymbol{LIN}$: number of solutions of linear system,
$\boldsymbol{CPU}$: CPU-time

3. Numerical results

In this section, we present some numerical results obtained with two examples. As the exact solution is unavailable, we compare the calculated solutions with a reference solution obtained using a very fine mesh.

3.1. Example 1

Let's consider a water flow on a flat bottom, i.e., bottom $B \equiv 0$, where the initial conditions are

$$h(x, 0) = 1 + e^{-x^2}, q(x, 0) = 0, a \leq x \leq b$$

and boundary conditions are [13]:

$$h(a, t) = h(b, t) = 1, q(a, t) = q(b, t) = 0$$

For the numerical simulation, the computational domain is $[-8; 8]$

Figure 1 shows the water height and the discharge profile with $n = 300$ and $nr = 2000$ points grid at $t = 0$.

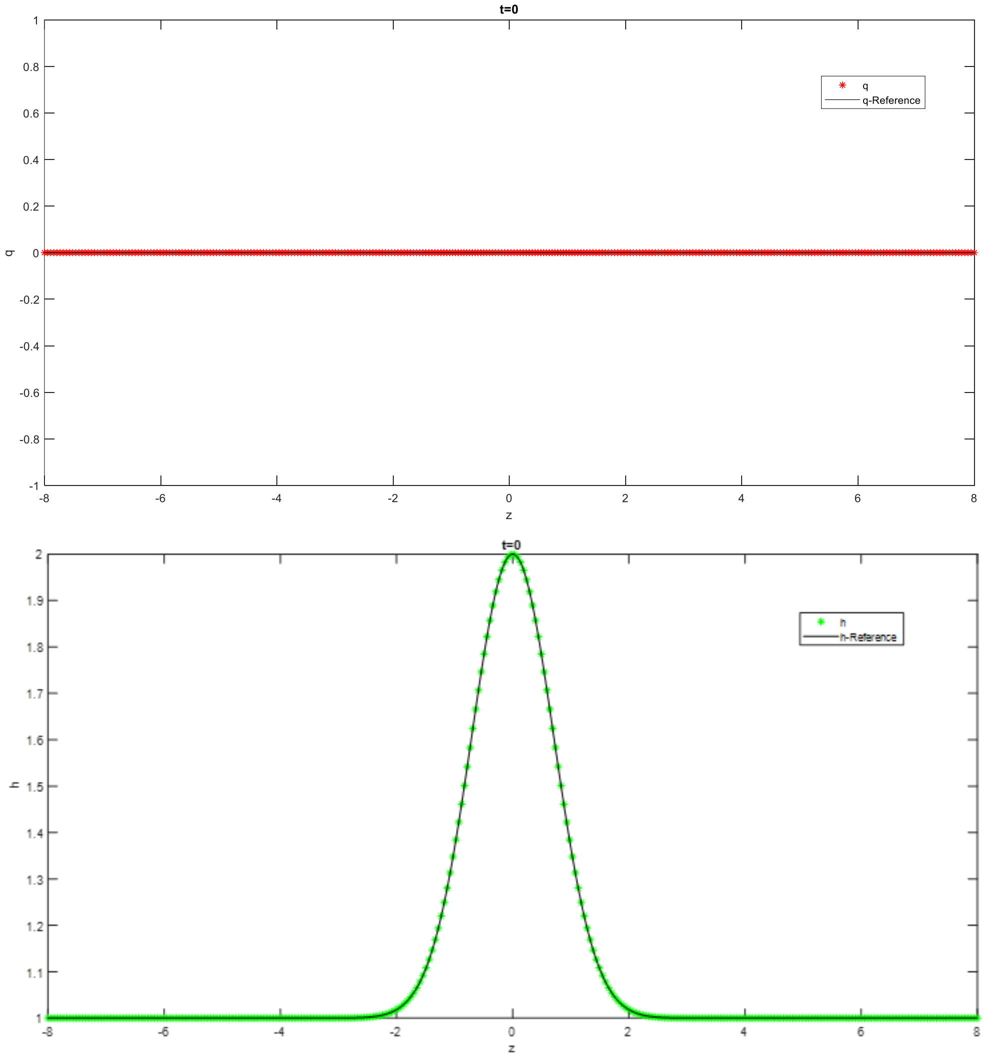
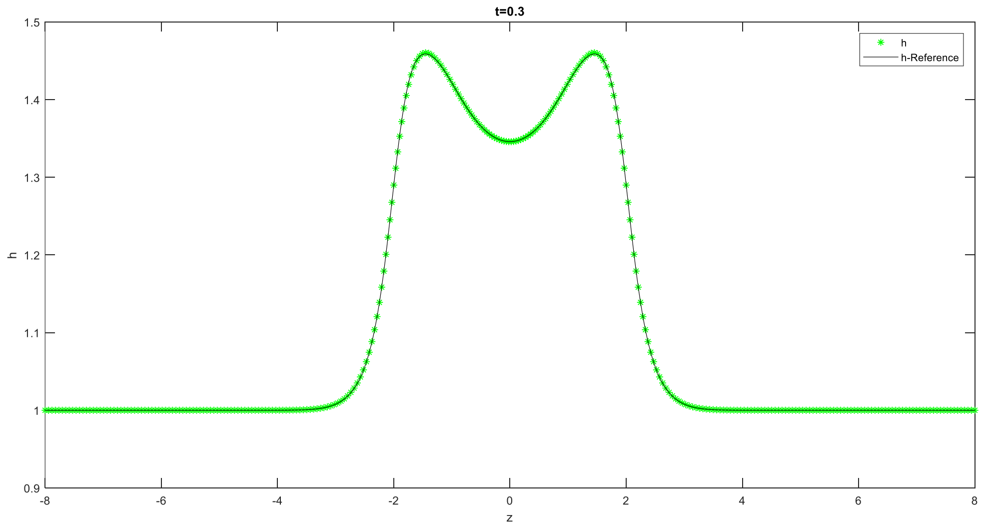


Figure 1: Numerical and reference solution at $t = 0$.



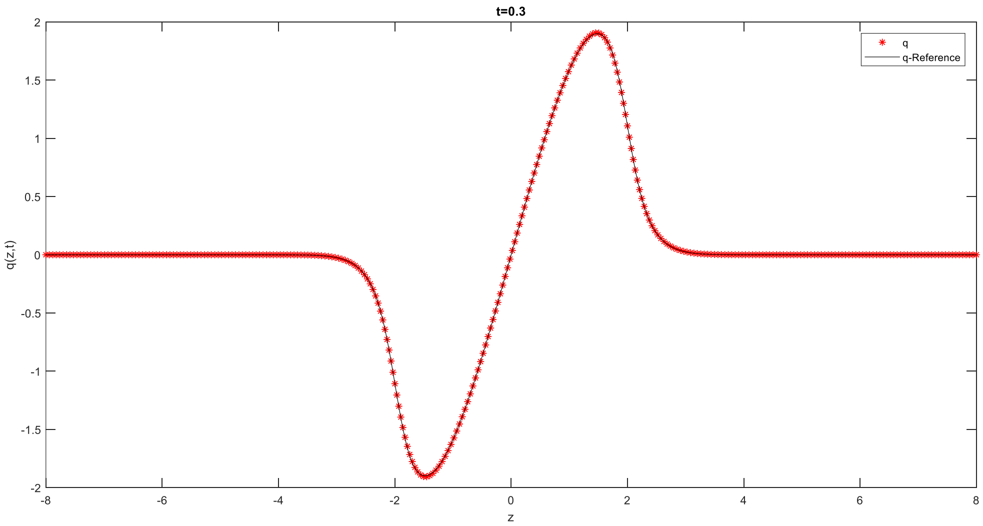


Figure 2: Comparison of solutions obtained with a moving grid for $n = 300$ *nodes and those a fixed uniform grid with* $nr = 2000$ *nodes for* h, q *at time* $t = 0.3$

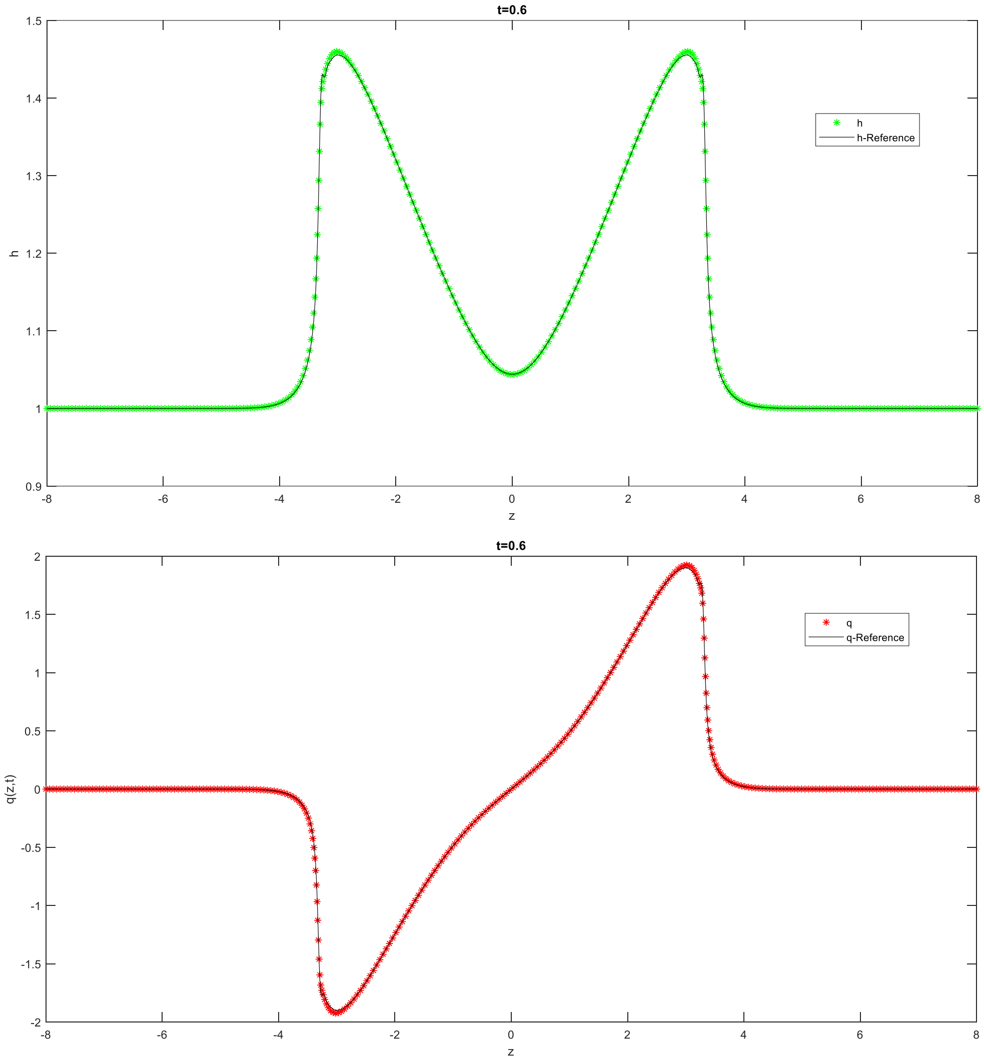


Figure 3: Comparison of solutions obtained with a moving grid for $n = 300$ *nodes and those a fixed uniform grid with* $nr = 2000$ *nodes for* h, q *at time* $t = 0.6$

Figures 2 and 3 shows the computed h and q at t = 0.3 and t=0.6, where the black solid line indicates the reference solution. From Figures, it can be clearly shown that, the moving grid with a small number of nodes, the solution is better and numerically stable.

Table 1: Computational statistics of shallow water system.

		$Suc.St$	$Fail.at$	$Fun.ev$	$Part.der$	$LU.dec$	$Sol.lin$	$CPU.t$
$t = 0.3$	$n = 300$	66	10	336	10	28	136	28.0015
	$nr = 2000$	24	1	64	1	7	42	129.1544
$t = 0.6$	$n = 300$	95	25	656	21	48	236	25.4024
	$nr = 2000$	67	11	190	1	17	168	55.5749

Table 1 shows that numerical results are satisfactory compared with those obtained by a very large number of nodes for a fixed grid.

3.2. Example 2

The secund case is an example with a parabolic bottom topography [14]. This problem simulates a flow over a bump. The initial conditions are given by:

$$h(x,0) = \begin{cases} 0.13 + 0.05(x-10)^2, & if\ 8 < x < 12 \\ 0.33 & othewise \end{cases}$$

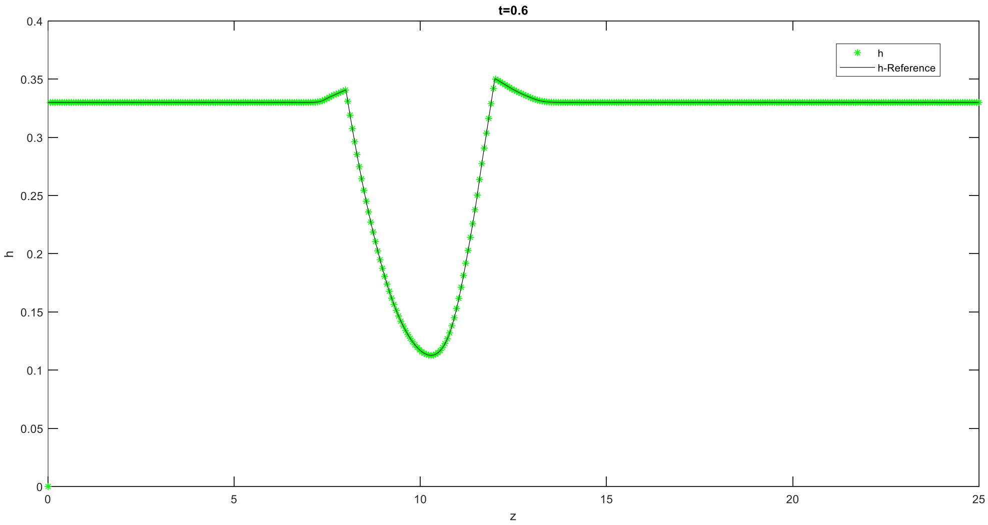
$$q(x,0) = 0.18$$

$$B(x,0) = \begin{cases} 3 - (x-10)^2, & if\ 8 < x < 12 \\ 2.8 & otherwise \end{cases}$$

The boundary conditions are given by:

$$q(a,t) = 0.18;\ h(b,t) = 0.33$$

The axis of the channel is the interval $[0, 25]$, we compare the numerical solutions with $n = 400$ to the reference one obtained by using a very fine mesh $nr = 2000$ and $nr = 4000$ cells.



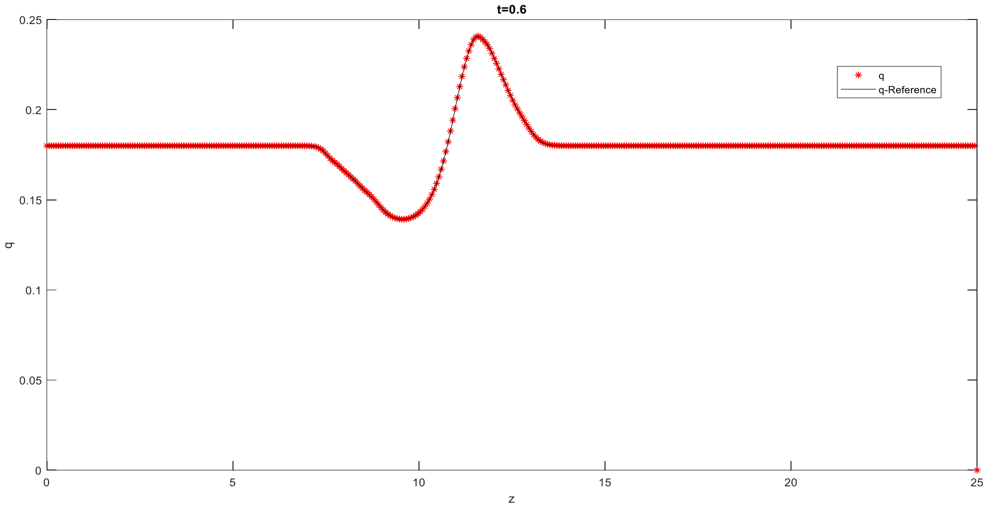


Figure 4: Comparison of solutions obtained with a moving grid for $n = 400$ nodes and those of a uniform fixed grid for $nr = 2000$ nodes at $t = 0.6$

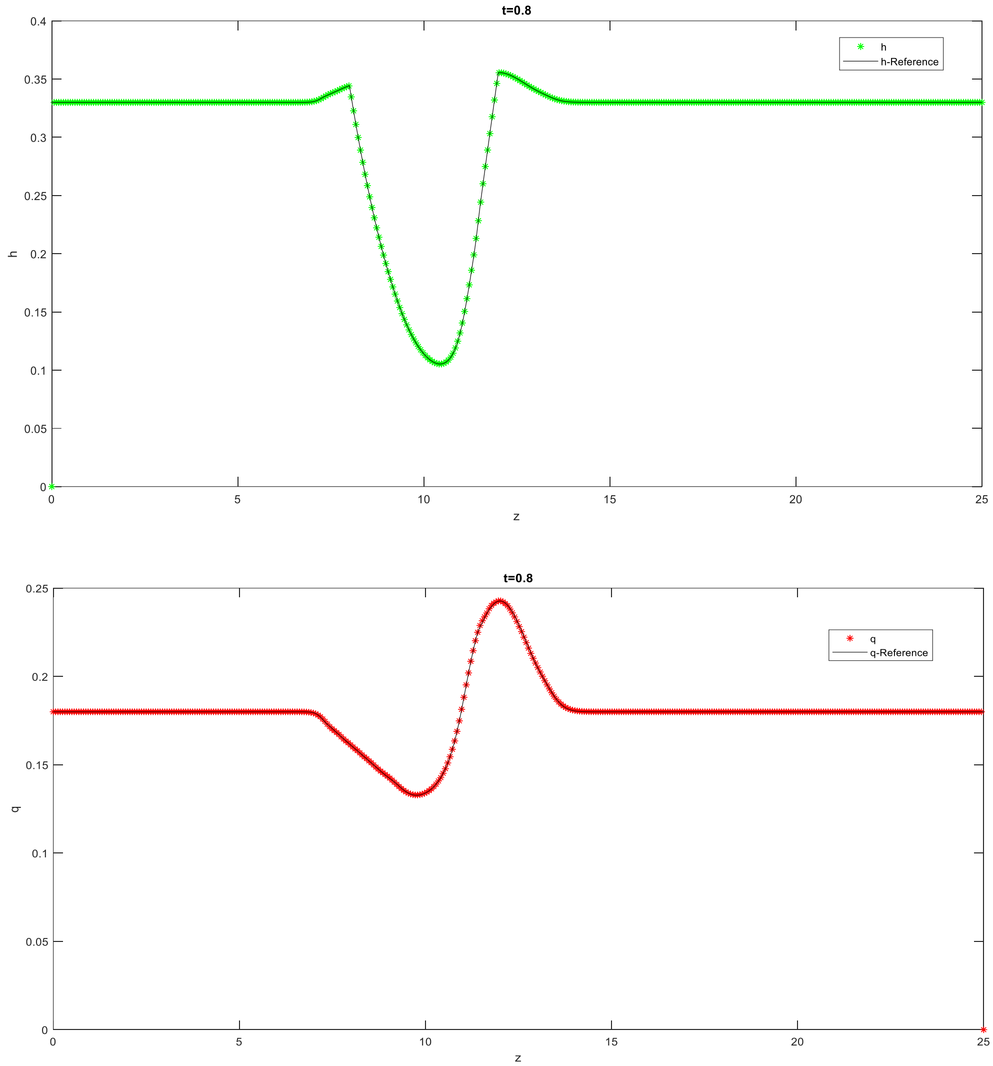


Figure 5: Comparison of solutions obtained with a moving grid for $n = 400$ nodes and those of a uniform fixed grid for $nr = 4000$ nodes at $t = 0.8$

Figure 4 and 5 shows that the numerical results are in very good agreement with reference solution.

Table 2: Computational statistics of shallow water system

		$Suc.St$	$Fail.at$	$Fun.ev$	$Part.der$	$LU.dec$	$Sol.lin$	$CPU.t$
$t = 0.6$	$n = 400$	34	0	70	1	12	50	12.0923
	$nr = 2000$	19	0	46	1	6	28	64.0271
$t = 0.8$	$n = 400$	37	0	76	1	13	56	9.0205
	$nr = 4000$	24	1	75	2	9	40	759.6754

We compute the numerical solution using $n = 400$ points in the interval $[0, 25]$ and compare the results with the reference solution computed on a fine grid with $nr = 2000$, $nr = 4000$ points at $t = 0.2$ and $t = 0.8$. As can be observed, these numerical results are good when moving node are used.

4. Conclusion

The numerical schemes studied are an excellent alternative method to approximate time-dependent partial differential equations. The results of the proposed scheme based on moving grid techniques in this paper, shown that, numerical solutions obtained are in a good agreement with the reference solutions. Satisfactory numerical accuracy and efficiency properties are observed.

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