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Fixed Point Property for Asymptotically Nonexpansive Mappings on a Large Class of Closed, Bounded and Convex Subsets in Köthe-Toeplitz Duals of Certain Generalized Difference Sequence Spaces

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Abstract In 1970, Cesàro sequence spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. Later, Et and Tripathy et. al. generalized the space introduced by Orhan for any $m \in \mathbb{N}$. Moreover, in 1989, Çolak obtained new types of sequence spaces by generalizing Kızmaz's idea and using Çolak's structure, Et and Esi, in 2000, obtained generalized difference sequences. In fact, they found the corresponding Köthe-Toeplitz duals and examined geometric properties for those spaces. We will be interested in their generalizations and we study Kaczor and Prus analogy for the spaces they introduced. We recall that in 2004, Kaczor and Prus saw that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically nonexpansive mappings. In the present study, we aim to discuss the analogous results for Köthe-Toeplitz duals of certain generalized difference sequence spaces studied by Et and Esi. We show that there exists a very large class of closed, bounded, convex subsets in those spaces with fixed point property for affine asymptotically nonexpansive mappings.

Keywords Fixed point property, nonexpansive mapping, Cesàro Difference Sequences, Köthe-Toeplitz dual
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1. Introduction and preliminaries

We say that a Banach space $(X, \|.\|)$ has the fixed point property for non-expansive mappings if every non-expansive self mappings defined on any non-empty closed, bounded and convex subset of the Banach space has a fixed point. Here we note that if C is a subset of the Banach space, then $\mathrm{T}: \mathrm{C} \to \mathrm{C}$ is said to be a nonexpansive mapping if $\|\mathrm{T}(\mathrm{x}) - \mathrm{T}(\mathrm{y})\| \leq \|\mathrm{x} - \mathrm{y}\|$, for all $\mathrm{x}, \mathrm{y} \in \mathrm{C}$. Researchers have been interested in checking if a nonreflexive Banach space can be renormed to have the fixed point property to see how the fixed point property is related with reflexivity. In fact, the first example of a nonreflexive Banach space which can be renormable such that its renorming becomes to have the fixed point property was given by Lin [17]. Lin got his result for the Banach space of absolutely summable scalar sequences, ℓ^1. Then, many have wondered the analogous result for the Banach space of scalar sequences converging to 0, c_0 as another well known classical non-reflexive Banach space sharing many common properties but the answer still remains open. Moreover, many have wondered the analogous result for the corresponding function space which yields ℓ^1 when the counting measure is used for the Banach space of Lebesgue integrable functions on [0,1], $L_1[0,1]$ and then the answer was given by Maria and Hernandes Lineares [18] under affinity condition. One can note that all these works have implemented the ideas of Goebel and Kuczumow [13] who showed that there exists very large class of non-weakly compact, closed, bounded and convex subsets of ℓ^1 respect to weak* topology of ℓ^1 with fixed point property for non-expansive mappings. In fact, Kaczor and Prus [14] wanted to generalize this result by investigating if the same could be done for asymptotically nonexpansive mappings. Then, as their result, they proved that there exists a large class of closed, bounded, convex subsets in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings. Furthermore, Everest, in his Ph.D. thesis [10], written under supervision of Chris Lennard, considered large classes in ℓ^1 with fixed point property for affine asymptotically non-expansive mappings by generalizing Kaczor and Prus' work.

In this study, we aim to discuss the analogous results for Köthe-Toeplitz duals of certain generalized difference sequence spaces studied by Et and Esi [9]. We show that there exists a very large class of closed, bounded, convex subsets in those spaces with fixed point property for affine asymptotically nonexpansive mappings. Thus, first we will recall the definition of Cesàro sequence spaces introduced by Shiue [21] in 1970 and next we will give Kızmaz's construction in [15] for difference sequence spaces since the dual space we work on is obtained from the generalizations of Kızmaz's idea which are derived differently by many researchers such as Çolak [6], Et [7], Et and Çolak [8], Et and Esi [9], NgPeng-Nung and and LeePeng-Yee [19], Orhan [20], and Tripathy et. al. [22]. But we need to note that Et and Esi's work [9] and the further study [8] by Et and Çolak used the new type of difference sequence definition from Çolak's work [6].

First we recall that Shiue [21] in 1970 introduced the Cesàro sequence spaces written as

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\} \text{ such that } \ell^p \subset \mathrm{ces}_p$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\} \text{ such that } \ell^\infty \subset ces_\infty$$

where $1 \leq p < \infty$. Their topological properties have been investigated and it has been seen that for $1 < p < \infty$, ces_p is a seperable reflexive Banach space. Moreover, Furthermore, many researchers such as Cui, Cui-Hudzik-Li and Cui-Meng-Pluciennik, in [3], [5] and [4] respectively, were able to prove that for $1 < p < \infty$, Cesàro sequence space ces_p has the fixed point property. Easiest way to show that was due to both reflexivity by the fact the space has normal structure when $1 < p < \infty$ (using the fact via Kirk [16]) and the space having the weak fixed point property because of its Garcia-Falset coefficient is less than 2 (see for example [11]). A good referance about fixed point theory results for Cesàro sequence spaces can be a survey in [2].

After the introduction of Cesàro sequence spaces, in 1981, Kızmaz [15], denoting by $\ell^\infty(\triangle)$, $\mathrm{c}(\triangle)$ and $\mathrm{c}_0(\triangle)$, introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences, respectively. Here $\triangle$ represented the difference operator applied to the sequence $x = (x_n)_n$ with the rule given by $\triangle x = (x_k - x_{k+1})_k$. Kızmaz studied then Köthe-Toeplitz Duals and topological properties for them.

As earlier it was stated, Çolak was one of the researchers generalizing Kızmaz's [15] ideas. In his work [6], Çolak obtained the generalized version of the difference sequence space in the following way by picking an arbitrary sequence of complex values $v = (v_n)_n$. The new difference operator is denoted by $\triangle_v$ and a sequence $x = (x_n)_n$ the difference sequence is written as $\triangle_v x = (v_k x_k - v_{k+1} x_{k+1})_k$. Then, in their study, Et and Esi [9] define a generalized difference sequence space as below.

$$\triangle_v (\ell^\infty) = \{ x = (x_n)_n \subset \mathbb{R} | \triangle_v x \in \ell^\infty \},$$
$$\triangle_v (\mathrm{c}) = \{ x = (x_n)_n \subset \mathbb{R} | \triangle_v x \in \mathrm{c} \},$$
$$\triangle_v (\mathrm{c}_0) = \{ x = (x_n)_n \subset \mathbb{R} | \triangle_v x \in \mathrm{c}_0 \}.$$

In fact, Et and Esi [9] further generalized the above difference sequence spaces and in 2004, Bektaş, Et and Çolak [1] not only found the Köthe-Toeplitz duals for them but also obtained the duals for the generalized types of Et and Esi's. In this study, we will only care the one we call the basic type whose norm is defined by

$$\|x\|_v^* = |v_1 x_1| + \sup_k |v_k x_k - v_{k+1} x_{k+1}|.$$

Then the corresponding Köthe-Toeplitz dual was obtained as in [1] and [9] such that it is written as below:

$$U_1 := \{a = (a_n)_n \subset \mathbb{R} | (n{v_n}^{-1} a_n)_n \in \ell^1\} = \left\{ a = (a_k)_k \subset \mathbb{R} : \ \|a\| = \sum_{k=1}^{\infty} k|{v_k}^{-1}| \, |a_k| < \infty \right\}.$$

Note that $U_1 \subset \ell^1$.

We will need the below well-known preliminaries before giving our main results. [12] may be suggested as a good reference for these fundamentals.

Definition 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and C is a non-empty closed, bounded, convex subset.*

1. If $T: C \to C$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in C$, $T\big((1-\lambda)x + \lambda y\big) = (1-\lambda)T(x) + \lambda T(y)$ then T is said to be an affine mapping

2. If $T: C \to C$ is a mapping such that $\| T(x) - T(y) \| \leq \| x - y \|$, for all $x, y \in C$ then T is said to be a nonexpansive mapping

Also, if for every nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for nonexpansive mappings [fpp(ne)]

3. If $T: C \to C$ is a mapping such that there exists a sequence of scalars $(k_n)_{n \in N}$ decreasingly approach to 1 and $\| T^n(x) - T^n(y) \| \leq k_n \| x - y \|$, for all $x, y \in C$ and for all $n \in \mathbb{N}$ then T is said to be an asymptotically nonexpansive mapping

Also, if for every asymptotically nonexpansive mapping $T: C \to C$, there exists $z \in C$ with $T(z) = z$, then C is said to have the fixed point property for asymptotically nonexpansive mappings [fpp(ane)]

Remark 1.1. *In 1979, Goebel and Kuczumow [13] showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{x_n\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $y \in \ell^1$,*

$$r(y) = r(x) + \|y - x\|_1 \ \ where \ \ r(y) = \limsup_n \|x_n - y\|_1$$

We will call this fact Type equation here.

2. Main Result

In this section, we work on Kaczor and Prus analogy for a Banach space contained in the Banach space of absolute summable scalar sequences, ℓ^1. The space we consider is U_1, the Köthe-Toeplitz dual of a certain generalized sequence space, which was introduced by Et and Esi [9]. We show that there exists a very large class of closed, bounded and convex subsets of the space with the fixed point property for affine asymptoticallys non-expansive mappings.

Now, let us first consider the following class of closed, bounded and convex subsets for U_1, by the following example and then we will provide a theorem about that large class. We should note here that we will be using similar ideas to those in in the section 2 of Ph.D. thesis of Everest [10], written under supervision of Chris Lennard, where Everest firstly provides Goebel and Kuczumow's proofs in detailed.

Example 2.1. *Fix $b \in (0,1)$. Define a sequence $(f_n)_{n \in \mathbb{N}}$ by setting $f_1 := b v_1 \, e_1$ and $f_n := \frac{v_n e_n}{n}$ for all integers $n \geq 2$ where the sequence $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, a closed, bounded, convex subset $E = E_b$ of the Köthe-Toeplitz Dual U_1 can be defined as below.*

$$E := \left\{ \sum_{n=1}^{\infty} t_n \, f_n : \ \forall n \in \mathbb{N}, \quad t_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Theorem 2.1. *For any $b \in (0,1)$, the set E defined as in the example above has the fixed point property for affine asymptotically nonexpansive mappings.*

Proof. Fix $b \in (0,1)$. Let $T: E \to E$ be an affine asymptotically nonexpansive mapping. Then, since T is affine, by Lemma 1.1.2 in the Ph.D. thesis of Everest [10] written under supervision of Lennard, there exists a sequence $\left(x^{(n)}\right)_{n \in \mathbb{N}} \in E$ such that $\left\|Tx^{(n)} - x^{(n)}\right\| \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $x \in E$ such that $x^{(n)}$converges to x in weak* topology. Then, by Goebel Kuczumow analog fact ∴ given in the last part of the previous section, we can define a function $s: U_1 \to [0, \infty)$ by

$$s(y) = \limsup_n \left\|x^{(n)} - y\right\| \ \ , \ \forall y \in U_1$$

and so

$s(y) = s(y) + \|x - y\|$, $\forall y \in U_1$.

Now define the weak* closure of the set E as it is seen below.

$$W := \overline{E}^{w^*} = \left\{ \sum_{n=1}^{\infty} \beta_n f_n : \ each\ \beta_n \geq 0\ \ and\ \sum_{n=1}^{\infty} \beta_n \leq 1 \right\}$$

Since T is asymptotically nonexpansive mapping, there exists a decreasing sequence $(k_n)_{n\in\mathbb{N}} \in [1, \infty)$ converging to 1 such that $\forall x, y \in U_1$ and $\forall n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

Case 1: $x \in E$

Fix $q \in \mathbb{N}$ and take $k_0 = 1$. Then, we have $s(T^q x) = S(x) + \|T^q x - x\|$ and

$$s(T^q x) = \limsup_n \|T^q x - x^{(n)}\| \leq \limsup_n \|T^q x - T^q(x^{(n)})\| + \limsup_n \|T^q(x^{(n)}) - x^{(n)}\| \quad (2.1)$$

$$\leq \limsup_n k_q \|x - x^{(n)}\| + \limsup_n \sum_{j=1}^{q} \|T^j(x^{(n)}) - T^{j-1}(x^{(n)})\|$$

$$\leq k_q \limsup_n \|x - x^{(n)}\| + \limsup_n \sum_{j=1}^{q} k_{j-1} \|T(x^{(n)}) - x^{(n)}\|$$

$$= k_q S(x).$$

Therefore, $\|T^q x - x\| \leq S(x)(k_q - 1)$ and so by taking limit as $q \to \infty$, we have $\lim_q \|T^q x - x\| = 0$ but then since $\lim_q \|T^{q+1} x - Tx\| \leq \lim_q \|T^q x - x\|$, $\lim_q \|T^{q+1} x - Tx\| = 0$ and so $T^q x$ converges both Tx and x; thus, $Tx = x$ by the uniqueness of the limits.

Case 2: $x \in W \backslash E$.

Then, x is of the form $\sum_{n=1}^{\infty} \gamma_n f_n$ such that $\sum_{n=1}^{\infty} \gamma_n < 1\ and\ \gamma_n \geq 0,\ \forall n \in \mathbb{N}$.

Define $\delta := 1 - \sum_{n=1}^{\infty} \gamma_n$ and next define

$$h := (\gamma_1 + \delta) f_1 + \sum_{n=2}^{\infty} \gamma_n f_n.$$

Then, $\|h - f\| = \|b\delta e_1\| = \mathrm{b}\delta$.

Now fix $y \in E$ of the form $\sum_{n=1}^{\infty} \beta_n f_n$ such that $\sum_{n=1}^{\infty} \beta_n = 1$ with $\beta_n \geq 0,\ \forall n \in \mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k \in \mathbb{N}$ where $\xi_1 := b\ v_1$, and $\xi_n := n^{-1} v_n$ for all integers $n \geq 2$ such that for each $n \in \mathbb{N}$, $f_n = \xi_n e_n$.

Then,

$$\|y - x\| = \left\| \sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k \right\| = \left\| \sum_{k=1}^{\infty} \beta_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k \right\|$$

$$= \left\| \sum_{k=1}^{\infty} (\beta_k - \gamma_k) f_k \right\|$$

$$= \sum_{k=1}^{\infty} \left| (\beta_k - \gamma_k) \frac{k \xi_k}{v_k} \right| \geq b \sum_{k=1}^{\infty} |\beta_k - \gamma_k|$$

$$\geq \mathrm{b} \left| \sum_{k=1}^{\infty} (\beta_k - \gamma_k) \right|$$

$$= \mathrm{b} \left| 1 - \sum_{k=1}^{\infty} \gamma_k \right|$$

$$= \mathrm{b}\delta$$

Hence,

$$\|y - x\| \geq b\delta = \|h - x\|.$$

Next, we have the following.

$s(h) = s(x) + \|h-x\| \leq s(x) + \|T^q h - x\| = s(T^q h)$ but this follows

$$= \limsup_n \|T^q h - x^{(n)}\| \text{ then similarly to the inequality (2.1)}$$
$$\leq \limsup_n \|T^q h - T^q(x^{(n)})\| + \limsup_n \|x^{(n)} - T^q(x^{(n)})\|$$
$$\leq k_q \limsup_n \|h - x^{(n)}\| + \limsup_n \sum_{j=1}^{q} \|T^j(x^{(n)}) - T^{j-1}(x^{(n)})\|$$
$$\leq k_q \limsup_n \|h - x^{(n)}\| + \limsup_n \sum_{j=1}^{q} k_{j-1} \|T(x^{(n)}) - x^{(n)}\|$$
$$\leq k_q \limsup_n \|h - x^{(n)}\| + 0$$
$$= k_q s(h).$$

Hence, $s(h) \leq s(T^q h) \leq k_q s(h)$ and so taking limit as $q \to \infty$, we have $\lim_q s(T^q h) = s(h)$; that is, $\lim_q s(x) + \|T^q h - x\| = \lim_q s(x) + \|h-x\|$ which means $\lim_q \|T^q h - x\| = \|h-x\|$ (2.2)

Moreover, for any $y \in E$,

$$\|y-h\| = \left\|\sum_{k=1}^{\infty} \beta_k f_k - (\gamma_1+\delta) f_1 - \sum_{n=2}^{\infty} \gamma_n f_n\right\| = \left\|\sum_{k=2}^{\infty} (\beta_k - \gamma_k) f_k + (\beta_1 - \gamma_1 - \delta) f_1\right\|$$
$$\leq \left\|\sum_{k=2}^{\infty} (\beta_k - \gamma_k) f_k\right\| + \|(\beta_1 - \gamma_1 - \delta) f_1\| = \sum_{k=2}^{\infty} \left|(\beta_k - \gamma_k)\frac{k\xi_k}{v_k}\right| + \left|(\beta_1 - \gamma_1 - \delta)\frac{\xi_1}{v_1}\right|$$
$$\leq \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b|\beta_1 - \gamma_1 - \delta| = \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b\left|\beta_1 + \sum_{k=2}^{\infty} \beta_k - \sum_{k=2}^{\infty} \beta_k - \gamma_1 - 1 + \sum_{k=1}^{\infty} \gamma_k\right|$$
$$= \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b\left|\sum_{k=2}^{\infty} \gamma_k - \sum_{k=2}^{\infty} \beta_k\right|$$
$$\leq \sum_{k=2}^{\infty} |\beta_k - \gamma_k| + b\sum_{k=2}^{\infty} |\beta_k - \gamma_k| = (1+b)\sum_{k=2}^{\infty} |\beta_k - \gamma_k| = \frac{1+b}{1-b}(1-b)\sum_{k=2}^{\infty} |\beta_k - \gamma_k|$$
$$= \frac{1+b}{1-b}\left[b\delta - b\delta + (1-b)\sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right] = \frac{1+b}{1-b}\left[b\big(1-(1-\delta)\big) - b\delta + (1-b)\sum_{k=2}^{\infty} |\beta_k - \gamma_k|\right]$$
$$= \frac{1+b}{1-b}\left[b\big(1-(1-\delta)\big) + (1-b)\sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right]$$
$$= \frac{1+b}{1-b}\left[b\left(\sum_{k=1}^{\infty} \beta_k - \sum_{k=1}^{\infty} \gamma_k\right) + (1-b)\sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right]$$
$$\leq \frac{1+b}{1-b}\left[b\sum_{k=1}^{\infty} |\beta_k - \gamma_k| + (1-b)\sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right].$$

Hence,

$$\|y-h\| \leq \frac{1+b}{1-b}\left[b|\beta_1 - \gamma_1| + \sum_{k=2}^{\infty} |\beta_k - \gamma_k| - b\delta\right] = \frac{1+b}{1-b}[\|y-x\| - \|h-x\|].$$

Now, fix $\varepsilon > 0$ and recall that $b \in (0,1)$. Then, we can choose $\mu(\varepsilon) := \frac{1-b}{1+b}\varepsilon \in (0,\infty)$ such that for any $y = \sum_{k=1}^{\infty} \beta_k f_k \in E$,

$$\big|\|y-x\| - \|h-x\|\big| \leq \|y-x\| - \|h-x\| < \mu.$$

Then, $\|y-h\| < \frac{1+b}{1-b}\mu = \varepsilon$.

So for every $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon)$ such that if $\big|\|y-x\| - \|h-x\|\big| < \mu$ then $\|y-h\| < \varepsilon$ so this implies for any sequence $(\vartheta_n)_n$ in E with $\lim_n \|\vartheta_n - x\| = \|h-x\|$ implies $\lim_n \|\vartheta_n - h\| = 0$. But then since in (2.2) we obtained $\lim_{q} \|T^q h - x\| = \|h - x\|$, we have $\lim_{q} \|T^q h - h\| = 0$.

Furthermore,

$$\|h - Th\| \leq \lim_{q} \|T^q h - h\| + \lim_{q} \|T^q h - Th\| \leq k_1 \lim_{q} \|T^{q-1} h - h\| = 0$$

Hence, $Th = h$ and so E has fpp(ane) as desired.

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