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Research Article

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Bayes estimation of compound Rayleigh distribution parameter in the composite LINEX loss of symmetry

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Abstract Based on the compound LINEX symmetric loss function, this paper studies the Bayes estimation of compound Rayleigh distribution parameter , and makes a random numerical simulation test on rationality and optimality of the parameter Bayes estimation.

Keywords compound Linex symmetric loss function; compound Rayleigh distribution; Bayesian estimation.
MR (2000) Subject classification 62G05; 62N01

1 Introduction

Mostert et al. (1999) came up with compound Rayleigh distribution, it is widely used in the areas of survival analysis and life testing. Its distribution function and probability density function are:

$$F(x;\theta,\lambda) = 1-\lambda^{\theta}(\lambda+x^2)^{-\theta}, \quad x>0,\theta>0,\lambda>0. \tag{1-1}$$

$$f(x;\theta,\lambda) = 2\theta\lambda^{\theta}x(\lambda+x^2)^{-(\theta+1)}, \quad x>0,\theta>0,\lambda>0. \tag{1-2}$$

Suppose $(X_1,X_2,\cdots,X_n)$ is a *i.i.d* sample from population, then the likelihood function of θ under this sample is:

$$L(x|\theta) = \prod_{i=1}^{n} 2\theta\lambda^{\theta}x_i(\lambda+x_i{}^2)^{-(\theta+1)} = 2^n\prod_{i=1}^{n}\frac{x_i}{(\lambda+x_i{}^2)}\theta^n\lambda^{n\theta}\prod_{i=1}^{n}(\lambda+x_i{}^2)^{-\theta} = T_1\theta^n e^{-\theta T} \tag{1-3}$$

Where $T_1 = 2^n\prod_{i=1}^{n}\frac{x_i}{(\lambda+x_i{}^2)}$, $T=\sum_{i=1}^{n}\ln(\lambda+x_i{}^2)-n\ln\lambda$.

Composite LINEX symmetrical loss function was proposed by Zhang (2007),Its expression is as follows:

$$L(\theta,\delta) = L_a(\theta,\delta)+L_{-a}(\theta,\delta) = e^{-a(\theta-\delta)}+e^{a(\theta-\delta)}-2, \ a>0 \tag{1-4}$$

Scholars have also made some studies on the compound Rayleigh distribution.

Bekker et al. (2000) the composite Rayleigh distribution was used to study the survival time of cancer patients in clinical trials; Wang and Lan (2012) the estimation of the scale parameter of compound Rayleigh distribution are discussed based on complete samples; Li et al. (2020) with the square error loss function, the Empirical Bayes estimation for scaleparameter of compound Rayleigh distribution is studied by using probability density kernel estimation.

Based on the compound LINEX symmetric loss function, this paper studies the Bayes estimation of compound Rayleigh distribution parameter , and makes a random numerical simulation test on rationality and optimality of the parameter Bayes estimation.

2. Bayes estimation of parameter θ

Lemma 2.1 [2] Based on the compound LINEX symmetrical loss function of (1-4), for any prior distribution $\pi(\theta)$, Bayes estimation of θ is

$$\delta(x) = \frac{1}{2a}ln(E(e^{a\theta}|X)/E(e^{-a\theta}|X)) \tag{2-1}$$

Selection Эрланга distribution as prior distribution of Rayleigh distribution's parameter θ then the prior distribution density function is:

$$\pi(\theta;\alpha)=\theta^2\alpha e^{-\alpha\theta},\alpha>0,\theta>0. \quad (2\text{-}2)$$

As a result, Bayes estimation of Rayleigh distribution's parameter θ, can be obtained by the following theorem.

Theorem 2.1 Based on the compound LINEX symmetrical loss function of (1-4), for prior distribution $\pi(\theta;\alpha)$, Bayes estimation of Rayleigh distribution's parameter θ is:

$$\delta(x)=\frac{n+\alpha}{2a}ln\left(\frac{T+\beta+a}{T+\beta-a}\right). \quad (2\text{-}3)$$

Proof: The posterior density of reliability index θ is

$$H(\theta|x)=\frac{L(x|\theta)\pi(\theta;\alpha)}{\int_0^{+\infty}L(x|\theta)\pi(\theta;\alpha)d\theta}$$

$$=\frac{T_1\theta^n e^{-\theta T}\theta^2\alpha e^{-\alpha\theta}}{\int_0^{+\infty}T_1\theta^n e^{-\theta T}\theta^2\alpha e^{-\alpha\theta}d\theta}$$

$$=\frac{\theta^{(n+2)}e^{-(T+\alpha)\theta}}{\int_0^{+\infty}\theta^{(n+2)}e^{-(T+\alpha)\theta}d\theta}$$

$$=\frac{(T+\alpha)^{n+3}}{\Gamma(n+3)}\theta^{n+2}e^{-(T+\alpha)\theta}. \quad (2\text{-}4)$$

Where $T=\sum_{i=1}^n\ln(\lambda+x_i{}^2)-n\ln\lambda$.

So the posterior distribution obeys the $\Gamma(n+3,T+\alpha)$ distribution. That is to say:

$$E\left(e^{-a\theta}|X\right)=\int_0^{+\infty}e^{-a\theta}\frac{(T+\alpha)^{n+3}}{\Gamma(n+3)}\theta^{n+2}e^{-(T+\alpha)\theta}d\theta$$

$$=\frac{(T+\alpha)^{n+3}}{\Gamma(n+3)}\int_0^{+\infty}\theta^{n+2}e^{-(T+\alpha+a)\theta}d\theta$$

$$=\left(\frac{T+\alpha}{T+\alpha+a}\right)^{(n+3)}. \quad (2\text{-}5)$$

$$E\left(e^{a\theta}|X\right)=\int_0^{+\infty}e^{a\theta}\frac{(T+\alpha)^{n+3}}{\Gamma(n+3)}\theta^{n+2}e^{-(T+\alpha)\theta}d\theta$$

$$=\frac{(T+\alpha)^{n+3}}{\Gamma(n+3)}\int_0^{+\infty}\theta^{n+2}e^{-(T+\alpha-a)\theta}d\theta$$

$$=\left(\frac{T+\alpha}{T+\alpha-a}\right)^{(n+3)}. \quad (2\text{-}6)$$

Therefore, by lemma 2.1, the Bayes estimation of parameter θ is

$$\delta(x)=\frac{1}{2a}ln\left[\frac{\left(\frac{T+\alpha}{T+\alpha-a}\right)^{n+3}}{\left(\frac{T+\alpha}{T+\alpha+a}\right)^{n+3}}\right]=\frac{n+3}{2a}ln\left(\frac{T+\alpha+a}{T+\alpha-a}\right).$$

3. E-Bayes estimation of parameter θ

Definition 3.1 [6] Let $\delta(\alpha)$ is the Bayes estimation of parameter θ, and α is super parameter, and $\delta(\alpha)$ is a continuous function, then we call

$$\delta_E(x)=\int_D\delta(\alpha)\pi(\alpha)d\alpha \quad (3\text{-}1)$$

as E-Bayes estimation of parameter θ.

Where $\int_D\delta(\alpha)\pi(\alpha)d\alpha<+\infty$, D is the collection of all possible values of α, $\pi(\alpha)$ is the density function of parameter α.

According to the properties of $\pi(\theta;\alpha)$,when $0<\alpha<0.5$, $\pi(\theta;\alpha)$ is a decreasing function of parameter θ. So we can choose uniform distribution as its prior distribution:

$$\pi(\alpha)=U(0,0.5). \quad (3\text{-}2)$$

Theorem 3.1 For Lomax distribution, if the prior density function of parameter θ is $\pi(\theta;\alpha)=\theta^2\alpha e^{-\alpha\theta}$; the prior density function of super parameter α is a uniform distribution in D, then the E-Bayes estimation of parameter θ:

$$\delta_E(x)=\frac{2n+1}{4ac}(f(T,a,c)-g(T,a)-f_1(T,a,c)+g_1(T,a)) \quad (3\text{-}3)$$

Among them $f(T,a,c)=(T+a+c)ln\,(T+a+c)$, $g(T,a)=(T+a)ln\,(T+a)$,

$f_1(T,a,c) = (T-a+c)ln\,(T-a+c),\, g_1(T,a) = (T-a)ln\,(T-a)$.

Proof: Based on the definition of the E-Bayes estimation

$$\delta_E(x) = \int_D \delta(\alpha)\pi(\alpha)d\alpha$$

$$= \int_0^{0.5} \frac{n+3}{2\alpha} ln\left(\frac{T+\alpha+a}{T+\alpha-a}\right) * \frac{1}{0.5-0} d\alpha$$

$$= \frac{n+3}{a} \int_0^{0.5} ln\left(\frac{T+\alpha+a}{T+\alpha-a}\right) d\alpha$$

$$= \frac{n+3}{a} [(T+a+0.5)l\,n(T+a+0.5) - (T+a)l\,n(T+a) - (T-a+0.5)l\,n(T-a+0.5) + (T-a)l\,n(T-a)]$$

$$= \frac{n+3}{a} (f(T,a) - g(T,a) - f_1(T,a) + g_1(T,a)).$$

Therefore, the E-Bayes estimation of parameter θ is

$$\frac{n+3}{4a} (f(T,a) - g(T,a) - f_1(T,a) + g_1(T,a)).$$

4. Random numerical tests

In this section, we will design a stochastic simulation to test the rationality and superiority of $\delta(x)$ and $\delta_E(x)$ as well as their relations. Here, our random samples of Lomax distribution are obtained by θ=2.5 and $\lambda = 1$,when $a = 2, \alpha \in (0,1)$, then $\delta(x)$ and $\delta_E(x)$ is shown in the following table.

Table 4-1 Bayes estimation of parameter θ

α	estimator	N=100	N=500	N=1000	N=4000	N=5000	N=6000	N=10000	poor
0.05	$\delta(x)$	2.5950	2.5123	2.5055	2.49488	2.4988	2.4987	2.4965	0.0145
0.20	$\delta(x)$	2.5852	2.5104	2.5045	2.4946	2.4986	2.4985	2.4964	0.0126
0.35	$\delta(x)$	2.5754	2.5085	2.5036	2.4943	2.4984	2.4984	2.4963	0.0107
0.50	$\delta(x)$	2.5657	2.5066	2.5026	2.4941	2.4982	2.4982	2.4962	0.0088
0.65	$\delta(x)$	2.5560	2.5047	2.5017	2.4939	2.4980	2.4981	2.4961	0.0069
0.80	$\delta(x)$	2.5465	2.5029	2.5008	2.4936	2.4978	2.4979	2.4960	0.0051
0.95	$\delta(x)$	2.5370	2.5010	2.4998	2.4934	2.4977	2.4978	2.4959	0.0032
	$\overline{\delta(x)}$	2.5658	2.5066	2.5026	2.4941	2.4982	2.4982	2.4962	
poor	$\Delta\delta$	0.0658	0.0066	0.0026	0.0059	0.0018	0.0018	0.0038	0.0088

Table 4-2 E-Bayes estimation of parameter θ

estimator	N=100	N=500	N=1000	N=4000	N=5000	N=6000	N=10000	poor
$\delta_E(x)$	2.5819	2.5097	2.5042	2.4945	2.4985	2.4985	2.4964	0.0120

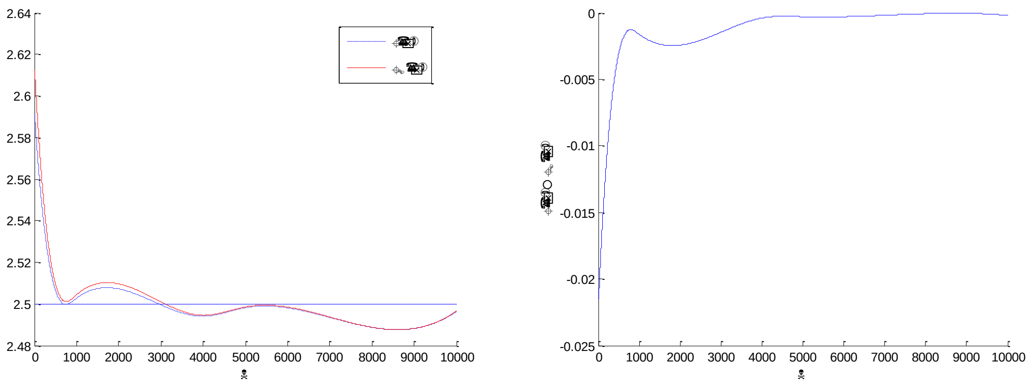

Figure 4-1: Comparison chart of $\delta(x)$ and $\delta_E(x)$

(1) From the view of robustness. As seen in Table 4-1 and Table 4-2, the range of $\delta(x)$ and $\delta_E(x)$ are 0.0088 and 0.0120 respectively, which are relatively small. Combining with Figure 4-1, we can clearly observe that $\delta(x)$ and $\delta_E(x)$ all tend to be stable while the sample size increases. So the robustness of $\delta(x)$ and $\delta_E(x)$ is good.

(2) From the view of accuracy. As seen in Table 4-1 and Table 4-2, $\delta(x)$ and $\delta_E(x)$ all tend to be actual values $\theta = 2.5$. From Figure 4-1, it can be clearly observed that two curves of $\delta(x)$ and $\delta_E(x)$ fluctuate around the actual value $\theta = 2.5$. Besides, we can find that the sample size between 5000 and 6000 is most appropriate.

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