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Transient buoyancy-induced flows of non-Newtonian fluids about a horizontal surface in an anisotropic porous medium

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Abstract The method of characteristics is used to exactly the problem of unsteady convection boundary layer flow along a horizontal surface embedded in an anisotropic porous medium saturated by a non-Newtonian fluid. The porous medium is anisotropic in permeability with its principal axes oriented in a direction that is oblique to the gravity vector. A step increase in wall temperature or in surface heat flux is considered. The modified Darcy power-law model proposed by Pascal [22, 23] and the generalized Darcy's law describing the saturating flow through the porous matrix, are used characterize the non-Newtonian fluid behavior. Scale analysis is applied to predict the order of magnitudes involved in the boundary layer regime. Analytical expressions are obtained for the boundary layer thickness and the mean Nusselt number in terms of the modified-Darcy Rayleigh number, the power-law exponent, the anisotropic permeability ratio, and the orientation angle of the principal axes. It is demonstrated that both the power-law exponent and the anisotropic properties have a strong influence on the heat transfer rate.

Keywords Transient convection, non-Newtonian fluid, boundary-layer flow, anisotropic porous medium

Introduction

The study of convective heat transfer in porous media is of great pratical importance and has thermal engineering applications in several areas such as geothermal engineering, cooling of electronic systems, porous journal bearings, thermal insulation systems, petroleum recovery, filtration processes ceramic processing, chromatography and ground water pollution. In a review article, Cheng [1] has discussed various works done in this field as applied to geothermal.

A cursory inspection of the existing references on convective external flow in porous media reveals that, in general, steady-state phenomena have been extensively studied, whereas unsteady phenomena have received relatively much less attention. The mechanism causing an unsteady flow may either act at the boundaries, and this may be through a change in one or more of the dependent variables, or it may be present within the fluid volume. Examples of the former include unsteadiness resulting from the movement of the system boundaries relative to the fluid and changing the upstream or the inlet conditions. Unsteadiness of the latter type results from changing the body forces, wall and internal energy generation rates, or the pressure gradients. More complex unsteadiness may include several of these effects simultaneously. Johnson and Cheng [2] have done a systematic analysis on the basis of the boundary-layer and Darcy approximations, regarding the possibility of similarity solutions for various wall temperature functions. These authors were the first to show that only very specific solutions exist for unsteady free convection about an inclined flat plate in a porous medium, and to summarize all the physical realizable similarity solutions. Raptis [3] has studied analytically unsteady two-dimensional free convective flow through a porous medium bounded by an infinite vertical plate, when the temperature of the plate is oscillating with the time about a constant non-zero mean value. The effects of the parameter of frequency on the velocity field were considered. Singh et al. [4] have extended theses analyses in solving the problem by asymptotic expansions development in powers of the frequency parameter, and discussed the effects of physical parameters on the velocity and temperature fields. Cheng and Pop [5] have used the method of integral relations to study the transient free convection about a vertical flat plate embedded in a porous medium and demonstrated the growth of the boundary-layer thickness for the case of a step increase in wall temperature.

In all the studies discussed earlier, the fluid saturating the porous medium was assumed to be Newtonian. However, in several of the engineering applications listed at the beginning of this section (such as oil recovery, food processing, the spreading of contaminants in the environment and in various processes in the chemical materials industry) the fluid saturating the porous matrix is necessarily Newtonian. For example, in the literature, the number of existing works in the limit of thermal convection in a porous medium saturated with a non-Newtonian fluid driven by temperature gradients alone is very limited. To this end, Chen and Chen [6] studied numerically the problem of boundary layer free convection about an isothermal vertical plate in a porous medium saturated by a power -law index fluid. Poulikakos and Spatz [7] investigated the effect of non-Newtonian natural convection at a melting front in a permeable matrix. Their results documented the dependence of the local heat transfer rate at the melting front on the type of power-law fluid saturating the porous matrix. Recently, Gorla et al. [8] presented a no similar boundary layer analysis for the problem of mixed convection in power-law type non-Newtonian fluids along horizontal surfaces with variable temperature distribution. A discussion of their work is provided for the effect of viscosity index on the surface heat transfer rate. Also, Gorla and Kumari [9] have presented a similarity solution for the problem of free convection boundary layer in power-law type non-Newtonian fluid along a horizontal plate with variable wall temperature or heat flux distribution. Numerical solutions were obtained for the flow and temperature fields for cases of variable surface temperature and variable surface heat flux and for the viscosity index.

Moreover, in all the above studies the porous media were assumed to be isotropic whereas, in several applications, the porous materials are anisotropic. Despite this fact, natural convection in such anisotropic porous media has received relatively little attention. The effects of an anisotropic permeability on thermal convection in a porous medium began with the investigation of Castinel and Combarnous [10], concerning the onset of motion in a horizontal layer heated from below, and continued with the works of Epherre [11], Kvernvold and Tyvand [12] and Nilsen and Storesletten [13]. Natural convection within enclosures heated from the side was investigated by Kimura et al [14] and Ni and Beckermann [15], for the case when one of the principal axes of anisotropy of permeability is aligned with gravity and by Zhang [16], Degan et al. [17] and Degan and Vasseur [18] when the principal axes are inclined with respect to gravity. It was demonstrated by these authors that the effects of the anisotropy considerably modify the convective heat transfer. Recently, the effects of anisotropy on the boundary-layer free convection over an impermeable vertical plate, for the case when one of the principal axes of anisotropy is along the plate, were investigated by Ene [19], using the method of integral relations. It was concluded that, if the permeability in the direction normal to the plate is greater than the permeability along the plate, then there is an increase in the temperature field. This investigation was extended by Vasseur and Degan [20] for the case when the porous medium is anisotropic in permeability with its principal axes oriented in a direction that is oblique to the gravity vector. Within the framework of boundary-layer approximations, similarity solutions are obtained for the case where wall temperature varies as a power function of distance from the leading edge. Solving numerically the full governing equations by using a finite-difference procedure, these authors demonstrated that the anisotropic parameters greatly influence the local heat transfer rates. Recently, Degan et al. [21] have investigated transient free convection boundary layer flow along a vertical surface embedded in an anisotropic porous medium saturated by a non-Newtonian fluid. Considering a step increase in wall temperature or surface heat flux, and using the method of characteristics, it was seen that both the power-law index and the anisotropic properties of the porous matrix have a strong influence on the leant transfer rate.

The present paper describes an analytical procedure for obtaining an exact solution for unsteady natural convection from a horizontal plate embedded in an anisotropic porous medium saturated by a non-Newtonian fluid. A step increase in wall temperature or in surface heat flux is considered. The porous medium is anisotropic in permeability with its principal axes oriented in a direction that is oblique to the gravity vector. Combining the modified Darcy power-law model proposed by Pascal [22,23] and the generalized Darcy's law proposed by Bear [24], a characterization of the saturating flow through the porous matrix is used to describe the non-Newtonian fluid behavior. In the large Rayleigh number limit, the boundary layer equation are solved analytically upon introducing a scale analysis to predict the order-of- magnitudes involved in the boundary layer regime.

2. Mathematical Formulation

We consider here the problem of unsteady heating of a horizontal impermeable plate embedded in a saturated porous medium characterized by an anisotropic permeability. The x and y axes are aligned with the horizontal and the vertical respectively. The saturating fluid is a non-Newtonian fluid of power-law behavior and the porous medium is at a uniform temperature T_∞.

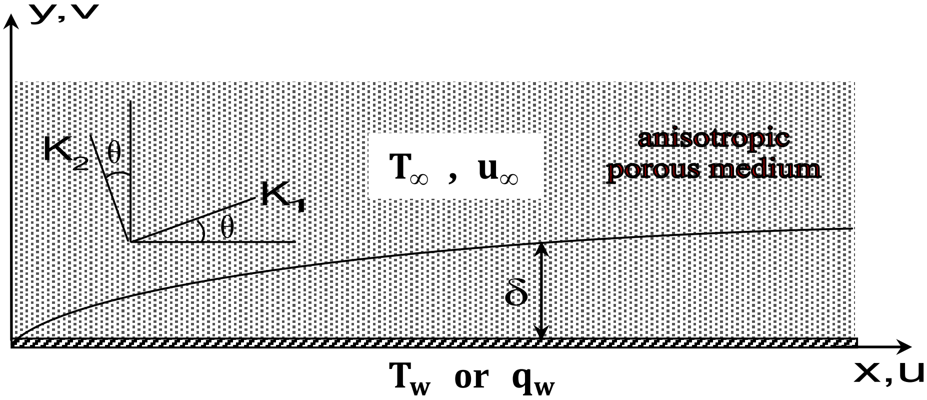

Figure 1: Physical model and coordinate system

At $t = 0$, the temperature of the surface is suddenly increased to the constant value T_w. The anisotropy of the porous medium is characterized by the anisotropy ratio $K^* = K_1/K_2$ and the orientation angle θ, defined as the angle between the horizontal direction and the principal axis with the permeability K_2. It is assumed that the fluid and the porous medium are everywhere in local thermodynamic equilibrium. The pressure and the temperature are such that the fluid remains in the liquid phase. The thermophysical properties of the fluid are assumed constant, except for the density in the buoyancy term in the momentum equation (i.e., the Boussinesq approximation).

In accordance with previous reports given by Pascal [22,23] and following Bear [24], the model of laminar flow of a non-Newtonian power-law fluid through the porous medium, describing the generalized Darcy's law, can be written as follows.

$$\boldsymbol{V} = -\frac{\boldsymbol{K}}{\mu_a}\boldsymbol{\nabla} P \tag{1}$$

$$\mu_a = \epsilon(u^2 + v^2)^{(n-1)/2} \tag{2}$$

$$\varepsilon = \frac{2\mu}{8^{(n+1)/2}(\gamma\sqrt{K_1K_2})^{(n-1)/2}(1+3n)^n} \tag{3}$$

Many of the inelastic non-Newtonian fluid encountered in engineering processes are known to follow a power-law model in which the pressure drop is proportional to the mass flow rate.

In the above equation, $\boldsymbol{V}$ is the superficial velocity, γ the porosity of the porous medium, μ_a the apparent viscosity, μ the consistency index and n the power-law index. In the above model, the rheological parameters μ and n are assumed to be temperature independent.

The equations describing conservation of mass, momentum and energy for the present problem are, respectively

$$\nabla.\mathbf{V} = 0 \tag{4}$$

$$\mathbf{V} = -\frac{\mathbf{K}}{\mu_a}(\nabla P + \rho g) \tag{5}$$

$$\sigma\frac{\partial T}{\partial t}+\nabla.(\mathbf{V}\boldsymbol{T}-\alpha\boldsymbol{\nabla T})=\mathbf{0} \tag{6}$$

$$\rho=\rho_\infty[1-\beta(T-T_\infty)] \tag{7}$$

In the above equations, T is the local equilibrium temperature of the fluid and the porous matrix, g the gravitational acceleration, P the pressure, t the time, $\alpha=k/(\rho c_p)_f$ the thermal diffusivity (k the thermal conductivity of fluid/porous matrix combination. $(\rho c_p)_f$ the heat capacity of the fluid), $\sigma=(\rho c_p)_p/(\rho c_p)_f$ the heat capacity ratio, β the coefficient of thermal expansion of the fluid and ρ the density. The symmetrical second-order permeability tensor **K** is defined as

$$\mathbf{K}=\begin{bmatrix}K_1\cos^2\theta+K_2\sin^2\theta & (K_1-K_2)\sin\theta\cos\theta\\ (K_1-K_2)\sin\theta\cos\theta & K_2\cos^2\theta+K_1\sin^2\theta\end{bmatrix} \tag{8}$$

Eliminating the pressure term by taking the curl of equation (5) and making use of equation (4), we obtain a single momentum, which reads

$$a\frac{\partial u}{\partial y}+c\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)-b\frac{\partial v}{\partial x}=\frac{1}{\mu_a}\left[\frac{\partial\mu_a}{\partial x}(bv+cu)-\frac{\partial\mu_a}{\partial y}(au+cv)+K_1\,g\rho_\infty\beta\frac{\partial T}{\partial x}\right] \tag{9}$$

Where

$$\begin{cases}a=\cos^2\theta+K^*\sin^2\theta\\ b=\sin^2\theta+K^*\cos^2\theta\\ c=\dfrac{1}{2}(1-K^*)sin2\theta\end{cases} \tag{10}$$

3. Scale Analysis

In this section, as t increases, the convection effect increases and we consider the boundary layer regime for which most of the fluid motion is restricted to a thin layer δ along the horizontal plate. From the momentum equation, equation (9), it is clear that we may use the boundary-layer hypothesis only when the following conditions

$$a\frac{\partial u}{\partial y}\gg c\frac{\partial v}{\partial y} \tag{11}$$

$$a\frac{\partial u}{\partial y}\gg c\frac{\partial u}{\partial x} \tag{12}$$

$$a\frac{\partial u}{\partial y}\gg b\frac{\partial v}{\partial x} \tag{13}$$

$$K_1g\rho_\infty\beta\frac{\partial T}{\partial x}\gg(cu+bv)\frac{\partial\mu_a}{\partial x} \tag{14}$$

$$K_1g\rho_\infty\beta\frac{\partial T}{\partial x}\gg(au+cv)\frac{\partial\mu_a}{\partial y} \tag{15}$$

are satisfied. So, under the boundary-layer approximations, at large modified Darcy-Rayleigh number, the governing equation become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{16}$$

$$\frac{\partial}{\partial y}(u^n)=\frac{n}{a}\frac{K_1g\rho_\infty\beta}{\epsilon}\frac{\partial T}{\partial x} \tag{17}$$

$$\sigma\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2T}{\partial y^2} \tag{18}$$

Which are to the solved subject to the following initial condition

$$t=0:\qquad T(x,y,0)=0 \tag{19}$$

The boundary conditions associated with previous governing equations are

$$\begin{cases}y=0:\quad v=0,\quad T(x,0,t)=T_w\quad (a)\\ y=0:\quad v=0,\quad \dfrac{\partial T(x,0,t)}{\partial y}=-\dfrac{q_w}{k}\quad (b)\end{cases} \tag{20}$$

$$y\to\infty:\qquad u=0,\qquad T(x,\infty,t)=T_\infty \tag{21}$$

Following Bejan [25] and recognizing H and δ as the x and y scales, respectively, in the boundary layer of interest ($\delta \ll H$), the conservation equations, equations (16), (17) and (18), require the following balances:

$$\frac{u}{L} \sim \frac{v}{\delta} \tag{22}$$

$$a\frac{u}{\delta} \sim \frac{1}{\epsilon u^{n-1}} K_1 g \rho_\infty \beta \frac{\Delta T}{L} \tag{23}$$

$$\sigma\frac{\Delta T}{t},\ u\frac{\Delta T}{L},\ v\frac{\Delta T}{\delta}\ \sim\ \alpha\frac{\Delta T}{\delta^2} \tag{24}$$

Where $\Delta T = (T_w - T_\infty)$ or $[q_w/(k/\mathrm{L})]$ is the characteristic scale of temperature. It is noticed that the temperature drop across the boundary layer is of the order of one. In the next subsections, it will be distinguish the result of the scale analysis for two cases, according to the heating process of the wall.

3.1 Isothermal wall

Solving the balances between equations (22)-(24) for $\delta,\ u,\ v\ and\ t$, we obtain the following results

$$\delta \sim L(Ra_L)^{-1/(2n+1)} a^{1/(2n+1)} \tag{25}$$

$$u \sim \frac{\alpha}{L}(Ra_L)^{2/(2n+1)}\, a^{-2/(2n+1)} \tag{26}$$

$$v \sim \frac{\alpha}{L}(Ra_L)^{1/(2n+1)}\, a^{-1/(2n+1)} \tag{27}$$

$$t \sim \frac{\sigma L^2}{\alpha}(Ra_L)^{-2/(2n+1)}\, a^{2/(2n+1)} \tag{28}$$

Where the modified Darcy-Rayleigh number Ra_L based on the length of the plate, is defined as

$$Ra_L = \frac{K_1 \rho_\infty g \beta \Delta T L^n}{\epsilon\, \alpha^n} \tag{29}$$

Defining the stream function ø related to the velocity components by

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \tag{30}$$

Such that the continuity equation, equation (4), is automatically satisfied, the scale for the stream function can be obtained as follows:

$$\psi \sim\ \alpha\, {Ra_L}^{1/(2n+1)} a^{-1/(2n+1)} \tag{31}$$

The average Nusselt number, Nu_L defined as the heat transfer over the pure heat conduction through the horizontal plate, has the following scale:

$$Nu_L = \frac{hL}{k} \sim {Ra_L}^{1/(2n+1)} a^{-1/(2n+1)} \tag{32}$$

where $h = q/(T_w - T_\infty)$ is the local heat transfer coefficient, $q = -k(\partial T/\partial y)|_{y=0}$ the local surface heat flux at the heated plate.

For the special case of an isotropic porous medium,$(K^* = 1, i.e., a = 1)$,the scales above reduce to those predicted by Gorla and Kumari [9] while studying free convection in non-Newtonian fluids along a horizontal plate in a porous medium.

The conditions of validity of the present boundary layer analysis now will be discussed. These results are expected to be valid only when the vertical boundary-layer is slender ($\delta \ll L$), i.e., for $Ra_L \gg a$. Furthermore, from equations (11)-(15) and (25)-(27), and making use of the results of the above order-of-magnitude analysis developed in this section, the boundary layer hypothesis is valid only when the conditions :

$$\mathrm{b} \ll {Ra_L}^{2/(2n+1)} a^{(2n-1)/(2n+1)} \tag{33}$$

and

$$\mathrm{c} \ll {Ra_L}^{1/(2n+1)} a^{2n/(2n+1)} \tag{34}$$

are satisfied.

3.2 Wall with uniform heat flux

Solving the balances between equations (22)-(24) for, $\delta,\ u\ , v$ and t, we obtain the following results

$$\delta \sim L\, {R_L}^{-1/[2(n+1)]} a^{1/[2(n+1)]} \tag{35}$$

$$u \sim \frac{\alpha}{L} {R_L}^{1/(n+1)} a^{-1/(n+1)} \tag{36}$$

$$v \sim \frac{\alpha}{L} {R_L}^{1/[2(n+1)]} a^{-1/[2(n+1)]} \tag{37}$$

$$t \sim \frac{\sigma L^2}{\alpha} {R_L}^{-1/(n+1)} a^{1/(n+1)} \tag{38}$$

$$\psi \sim \alpha\, {R_L}^{1/[2(n+1)]} a^{-1/[2(n+1)]} \tag{39}$$

$$Nu_L = \frac{hL}{k} \sim {R_L}^{1/2(n+1)} a^{-1/2(n+1)} \tag{40}$$

Where the modified Darcy-Rayleigh number R_L, based on the heat flux applied at the plate, is defined as:

$$R_L = \frac{K_1 \rho_\infty g \beta L^{n+1} q_\omega}{\epsilon\, \alpha^n k} \tag{41}$$

Taking into account the previous scales obtained in this case, the condition of existence of the horizontal boundary-layer hypothesis formulated by ($\delta \ll L$) becomes $R_L \gg a$. Moreover, making use of equations (11)-(15) and (36)-(38). The boundary-layer hypothesis is valid only when the conditions:

$$\mathrm{b} \ll {R_L}^{1/(n+1)} a^{n/(n+1)} \tag{42}$$

and

$$\mathrm{c} \ll {R_L}^{1/2(n+1)} a^{(2n+1)/[2(n+1)]} \tag{43}$$

are satisfied .

4. Resolution

On the one hand, taking L, $L/Ra_L^{1/(2n+1)}$, $\alpha Ra_L^{2/(2n+1)}/L$, $\alpha Ra_L^{1/(2n+1)}/L$, ΔT and $\sigma L^2/[\,\alpha Ra_L^{2/(2n+1)}]$ as respective dimensional scales for length, velocities in x and y directions, temperature and time, concerning the case of isothermal wall and on the other hand, setting L, $L/R_L^{1/2[(n+1)]}$, $\alpha R_L^{1/(n+1)}/L$, $\alpha R_L^{1/[2(n+1)]}/L$, ΔT and $\sigma L^2/[\alpha R_L^{1/(n+1)}]$ as respective dimensional scales for length, velocities in x and y directions, temperature and time, for the case of wall with uniform heat flux, it is found that the dimensionless boundary-layer equations are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{44}$$

$$\frac{\partial}{\partial Y}(U^n) = \frac{n}{a}\frac{\partial \Theta}{\partial X} \tag{45}$$

$$\frac{\partial \Theta}{\partial \tau} + U\frac{\partial \Theta}{\partial X} + V\frac{\partial \Theta}{\partial Y} = \frac{\partial^2 \Theta}{\partial Y^2} \tag{46}$$

Integrating equation (45) from $Y = 0$ to $Y = \infty$ (region situated is the free stream), one can have

$$U = \left(\frac{n}{a}\right)^{1/n} \left[\frac{\partial}{\partial X}\int_0^\infty \Theta dY\right]^{1/n} \tag{47}$$

Making use of continuity equation, equation (44), the integration of equation of energy, equation (46), yields

$$\frac{\partial}{\partial \tau}\int_0^\infty \Theta\, dY + \frac{\partial}{\partial X}\int_0^\infty U\Theta\, dY = -\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=0} \tag{48}$$

Substituting equation (47) into equation (48) and rearranging the resulting expression, we obtain finally

$$\frac{\partial}{\partial \tau}\int_0^\infty \Theta\, dY + \left(\frac{n}{a}\right)^{1/n} \frac{\partial}{\partial X}\int_0^\infty \left[\frac{\partial}{\partial X}\int_0^\infty \Theta\, dY\right]^{1/n} \Theta\, dY = -\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=0} \tag{49}$$

The problem of unsteady natural convection in a porous medium about a vertical or a horizontal, semi-infinite flat plate with a step increase in wall temperature or surface heat flux, considered here, gives rise, as the classical problem of a viscous boundary- layer in a free fluid, to the singularity problem in passing from the initial stage when the leading edge is not felt to the steady state defined for large time. For small values of time, the solutions for velocity and temperature are independent of time, for large values of time the solutions are independent of time. The singularity value of time depends on the horizontal X. As pointed out by Ene and Polisevski [26], the heat transfer characteristics change suddenly from transient, one-dimensional heat conduction to steady two-dimensional natural convection. So, equation (49) is to be solved subject to the initial condition (19) which becomes

$$\tau = 0: \qquad \Theta(X, Y, 0) = 0 \tag{50}$$

and the dimensionless boundary conditions prevailing at the horizontal plate are:

$$\begin{cases} Y = 0: & \Theta(X, 0, \tau) = 1 \quad (a) \\ & \dfrac{\partial}{\partial Y}\Theta(X, 0, \tau) = -1 \quad (b) \end{cases} \tag{51}$$

In the following subsections, equation (49) will be differentially solved by considering the two kinds of boundary conditions (51) imposed in the present analysis.

4.1 Isothermal wall

Following Cheng and Pop [5], with the boundary condition (51a), we assume a temperature profile of the form

$$\Theta = erfc(\eta) \tag{52}$$

Where erfc is the complementary error function and ç is expressed by

$$\eta = \frac{Y}{A} Ra_L^{-1/(2n+1)} \tag{53}$$

Where $A = \delta/L$ is the dimensionless boundary-layer thickness. Substituting equations (52) and (53) into equation (49) and after integrating yields

$$\frac{\partial A}{\partial \tau} + \left[\frac{n}{a\sqrt{\pi}} Ra_L^{1/(2n+1)}\right]^{1/n} \frac{\partial}{\partial X}\left[A\left(\frac{\partial A}{\partial X}\right)^{1/n}\right] = \frac{2}{A} Ra_L^{-2/(2n+1)} \tag{54}$$

Subject to the initial and boundary conditions

$$\begin{cases} \tau = 0 & A(X, 0) = 0 & (a) \\ \tau \geq 0 & A(X, \tau\ ) = 0 \ \ at \ \ X = 0 & (b) \end{cases} \tag{55}$$

Equation (54) is a partial differential equation of the hyperbolic type in $A(X, \tau)$. It will be solved exactly by the method of characteristics. Solving by the method of characteristics, the differential system equivalent to equation (54) is expressed as

$$\frac{X^{[1/(2n+1)]} dX}{2\,\zeta_n^{-2}\left(\dfrac{2n+1}{n+1}\right) a^{-2/(2n+1)}} = d\tau = Ra_L^{[2/(2n+1])} \frac{AdA}{2} \tag{56}$$

which has the characteristics

$$X^{[1/(2n+1)]} dX = 2\,\zeta_n^{-2}\left(\frac{2n+1}{n+1}\right) a^{[-2/(2n+1)]} d\tau \tag{57}$$

where

$$\zeta_n = \left[\frac{\sqrt{\tau}(2)^n}{n+1}\left(\frac{2n+1}{n}\right)^{n+1}\right]^{[1/(2n+1)]} \tag{58}$$

On each characteristic, A is related by

$$Ra_L^{[2/(2n+1)]} AdA = 2d\tau \tag{59}$$

or

$$Ra_L^{[2/(2n+1)]}\,\zeta_n^{-2}\left(\frac{2n+1}{n+1}\right) a^{[-2/(2n+1)]} AdA = X^{[1/(2n+1)]}\, dX \tag{60}$$

depending on whether the characteristic intercepts the $\tau -$ or $X -$ axis. Integrating equation (59) with the initial condition (55(a)) gives

$$A = 2\sqrt{\frac{\tau}{Ra_L^{[2/(2n+1)]}}} \tag{61}$$

and solving equation (60) subject to be boundary condition (55(b)) yields

$$A = \zeta_n\, Ra_L^{[-1/(2n+1)]} a^{\,[1/(2n+1)]}\, X^{\,[(n+1)/(2n+1)]} \tag{62}$$

As shown by Cheng and Pop (5) and revisited by Ene and Polisevki [26] in their analyses when studying the same problem along a vertical surface in the case of Newtonian fluid saturating an isotropic porous medium, the expression for A changes from equation (61) to equation (62) along the limiting line characteristic

$$\tau = \tau_s = \frac{1}{4}\,\zeta_n^2\, a^{[2/(2n+1)]} X^{[2(n+1)/(2n+1)]} \tag{63}$$

so that, equation (63) is a straight line or a curved line which divides the $X - \tau$ plane into two regions: a lower region for which A is given by equation (61) and an upper region for which A is given by equation (62). That limiting line characteristic provides the limit time reached in steady-state regime τ_s expressed by equation (63).

Under these considerations, we have two expressions of temperature profile corresponding to the two regions. For $\tau < \tau_s$ (in the lower region), one can find

$$\Theta = erfc\left(\frac{Y}{2\sqrt{\tau}}\right) \tag{64}$$

And for $\tau > \tau_s$ (in the upper region), the temperature distribution is expresses as

$$\Theta = erfc\left\{\frac{Y}{\zeta_n\, a^{[1/(2n+1)]}} X^{-[(n+1)/(2n+1)]}\right\} \tag{65}$$

As expected, equation (64) is independent of X, and this explains that the solution represents the transient heat conduction in a semi-infinite porous medium while equation (65), however, is independent of τ, and its solution represents steady-state natural convection.

The local Nusselt number, Nu_x through the plate is defined, in physical variables, by

$$Nu_x = \frac{q_w\, x}{k(T_w - T_\infty)} \tag{66}$$

Making use of equations (52), (53), (61) and (62), one can rewrite q_w as

$$q_w = \frac{2k(T_w - T_\infty)}{AL\sqrt{\pi}} \tag{67}$$

which becomes, after substitution of equation (61) valid in the lower region where transient heat conduction is predominant, what follows.

$$Q_w = \frac{1}{\sqrt{\pi\tau}} \tag{68}$$

where $Q_w = q_w L/\left[k\Delta T Ra_L^{1/(2n+1)}\right]$ is the dimensionless local surface heat flux.

Taking into account equations (52), (53), (57) and (62), equation (59) becomes

$$q_w = \frac{2k(T_w - T_\infty)}{x\sqrt{\pi}}\,\frac{1}{\zeta_n}\left(\frac{Ra_x}{a}\right)^{1/(2n+1)} \tag{69}$$

valid in the upper zone where convection heat transfer is predominant (i.e., for $\tau > \tau_s$). So, substituting equation (69) into equation (66), we can express the local Nusselt number Nu_x as

$$Nu_x = \frac{2}{\zeta_n\sqrt{\pi}}\left(\frac{Ra_x}{a}\right)^{1/(2n+1)} \tag{70}$$

where $Ra_x = K_1\rho_\infty g\beta(T_w - T_\infty)x^n/(\epsilon\,\alpha^n)$ is the modified Darcy-Rayleigh number, based on the distance x from the edge of the plate.

On the other hand, the convective flow is described by the stream function which can be expressed from equations (30), (47), (52), (53) and (62) as

$$\Psi = \zeta_n^{\,(n+1)/n}\left[\frac{n(n+1)}{(2n+1)\sqrt{\pi}}\right]^{1/n}\left(\frac{Ra_x}{a}\right)^{1/(2n+1)} \tag{71}$$

Where $\Psi = \psi/\alpha$.

It is obvious that by setting $n = 1$ in the limit of the same problem in a steady regime with a Newtonian Fluid saturating the porous matrix assumed isotropic in permeability for which the value of the anisotropic parameter a equals to unity, the results presented in the present analysis are found to be in a good agreement with those obtained by Gorla and Kumari [9].

4.2. Wall with constant heat flux

The problem given by equation (49) with the conditions, equation (50) and (51b) may be treated for the case of constant plate heat flux in similar fashion. Seeking the analogous form as previously of the solution which satisfies conditions evoked in this case, one can find after algebraic manipulations what follows

$$\Theta = \left[\frac{A\sqrt{\pi}}{2} R_L^{1/[2(n+1)]}\right] erfc(\eta) \tag{72}$$

where erfc is the complementary error function and η is expressed by

$$\eta = \frac{Y}{A} R_L^{-1/[2(n+1)]} \tag{73}$$

substituting equation (72) and (73) into equation (49), the equation for the boundary-layer thickness is obtained by the relation :

$$\frac{\partial(A^2)}{\partial\tau} + \left[\frac{n}{a}\frac{R_L^{1/(n+1)}}{2}\right]^{1/n} \frac{\partial}{\partial X}\left[A^2\left(\frac{\partial(A^2)}{\partial X}\right)^{1/n}\right] = 2R_L^{-1/(n+1)} \tag{74}$$

The resolution of equation (74) subject to the initial and boundary conditions (55(a)) and (55(b)) by the method of characteristics yields the following line expressed by

$$\left(\frac{a}{n}\right)^{1/(n+1)} dX = d\tau = R_L^{1/(n+1)} A\, dA \tag{75}$$

In this case, on each characteristic, A is related by

$$R_L^{1/(n+1)} A\, dA = d\tau \tag{76}$$

or

$$\left(\frac{n}{a} R_L\right)^{[1/(n+1)]} AdA = dX \tag{77}$$

Integrating equations (76) and (77) subject to conditions (55(a)) and (55(b)), respectively, A is evaluated by

$$A = \sqrt{2\tau}\; R_L^{[-1/2(n+1)]} \tag{78}$$

and

$$A = \left(\frac{a}{nR_L}\right)^{[1/2(n+1)]} \sqrt{2X} \tag{79}$$

respectively, such that, the expression of A changes from equations (78) to (79) along the limiting line characteristic given by

$$\tau = \tau_s = \left(\frac{a}{n}\right)^{1/(n+1)} X \tag{80}$$

As expected, the $X - \tau$ plane is divided into two regions, a lower one for which A is calculated by equation (78) and an upper one for which A is calculated by equation (79). Then, the limit time τ_s corresponding to the steady-state regime is predicted here by equation (80).

In the other terms, one can derive the temperature profile valid for each region considered as follows.

In a lower region (for which $\tau < \tau_s$):

$$\Theta = \sqrt{\frac{\pi\tau}{2}}\, erfc\left(\frac{Y}{\sqrt{2\tau}}\right) \tag{81}$$

and in an upper region (for which $\tau > \tau_s$):

$$\Theta = \left(\frac{a}{n}\right)^{[1/2(n+1)]} \sqrt{\frac{\pi X}{2}}\, erfc\left\{\frac{Y}{\sqrt{2X}}\left(\frac{n}{a}\right)^{[1/2(n+1)]}\right\} \tag{82}$$

As in the previous case, we have a solution for transient heat conduction in a semi-infinite plane with a step increase in surface heat flux which is in good agreement with the temperature obtained from equations (81) and (72). The solution, equation (82), represents steady-state natural convection on the horizontal plate and it is also in good agreement with the similarity solution (see Gorla and Kumari [9]).

The heat flux the horizontal surface in the lower region for which the transient heat conduction prevails is then, after substitution of equation (78) into equation (67):

$$Q_w = \sqrt{\frac{2}{\pi\tau}} \tag{83}$$

Where $Q_w = q_w/(k\Delta T R_L^{[1/2(n+1)]}/L)$ is the dimensionless local heat flux.

Considering the situation dominated by convection pattern, when $\tau > \tau_s$, and taking into account equation (79) and equation (67), the local Nusselt number, Nu_x, defined by equation (66) is calculated here by the following expression

$$Nu_x = \sqrt{\frac{2}{\pi}}\left(\frac{nR_x}{a}\right)^{[1/2(n+1)]} \tag{84}$$

In the same way, from equations (30), (47), (72) (73) and (79) the stream function is calculated by the expression

$$\Psi = 2^{[(n+2)/(2n)]}\left(\frac{nR_x}{a}\right)^{[1/2(n+1)]} \tag{85}$$

where $\Psi = \psi/\alpha$ and $R_x = K_1\rho_\infty g\beta q_w x^{n+1}/(\epsilon k\alpha^n)$ is the modified Darcy-Rayleigh number, based on the distance x from the edge of the surface, for the case of a wall heated by a constant heat flux.

5. Results and Discussion

Figures 2 and 3 illustrate the effect of the power-law index on non-Newtonian fluid on the limiting line characteristic given by equations (63) and (80), respectively, and are expressed by the time taken to reach steady-state flow, τ_s, when the transient free convection in the porous medium occurs as a result of a step increase in wall temperature and a step increase in surface constant heat flux. In Figures 2 and 3, it is observed that, when anisotropic parameters are held constant, for example, for $\theta = 45°$, and $K^* = 0.1$, each limiting line characteristic corresponding to a fixed power-law index n is a line which divides, as expected, the $X - \tau$ plane into two regions, a lower one dominated by a pure conduction regime for $\tau < \tau_s$ and the upper one dominated by a convective heat transfer for $\tau \geq \tau_s$. Moreover, it is seen that, as the power-law index increases from $n = 0.6$ (corresponding to a shear-thinning fluid) to $n = 1.4$ attributed to a shear-thicknening fluid, the upper region becomes progressively larger. This behavior can be explained by the fact that the upper region where convection effect is considerable over the limiting time, the local heat transfer calculated for the case of an isothermal wall by equation (70) becomes$Nu_x/Ra_x^{0.45} = 0.454$ for $n = 0.6$ while $Nu_x/Ra_x^{0.26} = 0.59$ for $n = 1.4$, and when the wall is heated by a constant heat flux, the local heat transfer expressed by equation (84) becomes $Nu_x/Ra_x^{0.31} = 0.81$ for $n = 0.6$ while $Nu_x/Ra_x^{0.21} = 0.97$ for $n = 1.4$. So, convection motion is enhanced, taking place in the upper region which becomes more and more important than the lower one, as the power-law index n is made higher.

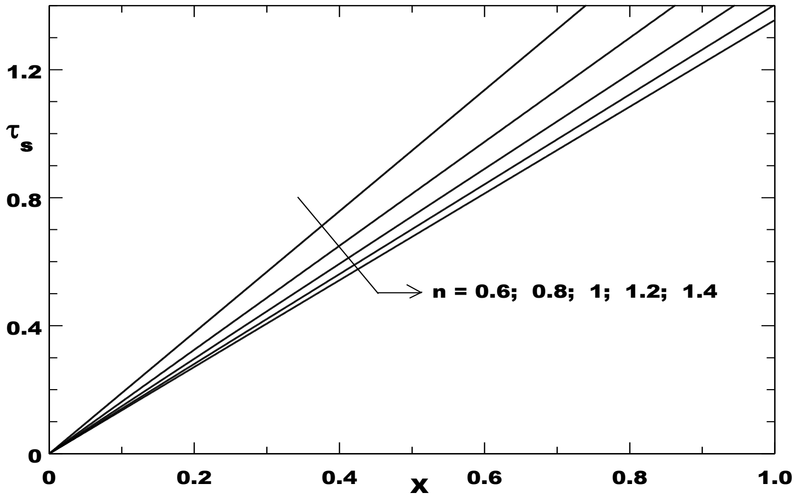

Figure 2: Effect of power-law index n on the limiting line characteristic for $\theta = 45°$, $K^* = 0.1$ and for step increase in wall temperature case

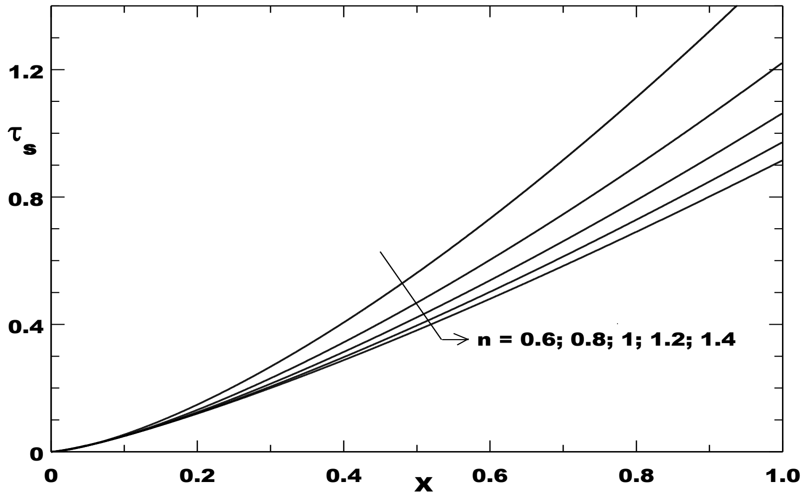

Figure 3: Effect of power-law index n on the limiting line characteristic for $\theta = 45°$, $K^* = 0.1$ and for step increase in wall heated by a constant heat flux case.

The limiting steady-state time characteristic τ_s is presented in Figures 4 and 5 as a function of the position X from the edge of the horizontal surface to investigate the effect of anisotropic permeability ratio K^* for $n = 1.2$ and $\theta = 30°$, when the wall is heated isothermally and by constant heat flux, respectively. It is observed that, for a given value of the distance X from the edge of the wall, the time taken by the heating process transfer to reach steady-state pattern for which convection occurs increases with an increase in permeability ratio K^*. So, convection becomes more and more considerable and occupies upper regions which become more and more large than the lowers regions, as K^* is made smaller. This trend comes from the fact that, according to equations (63) and (80), when the parameters n, θ and X are held constant, the limiting steady-state time τ_s when the wall is heated isothermally depends solely on $(K^*)^{0.99}$ and is proportional to the latter and this, in the particular case when $\theta = 90^0$.

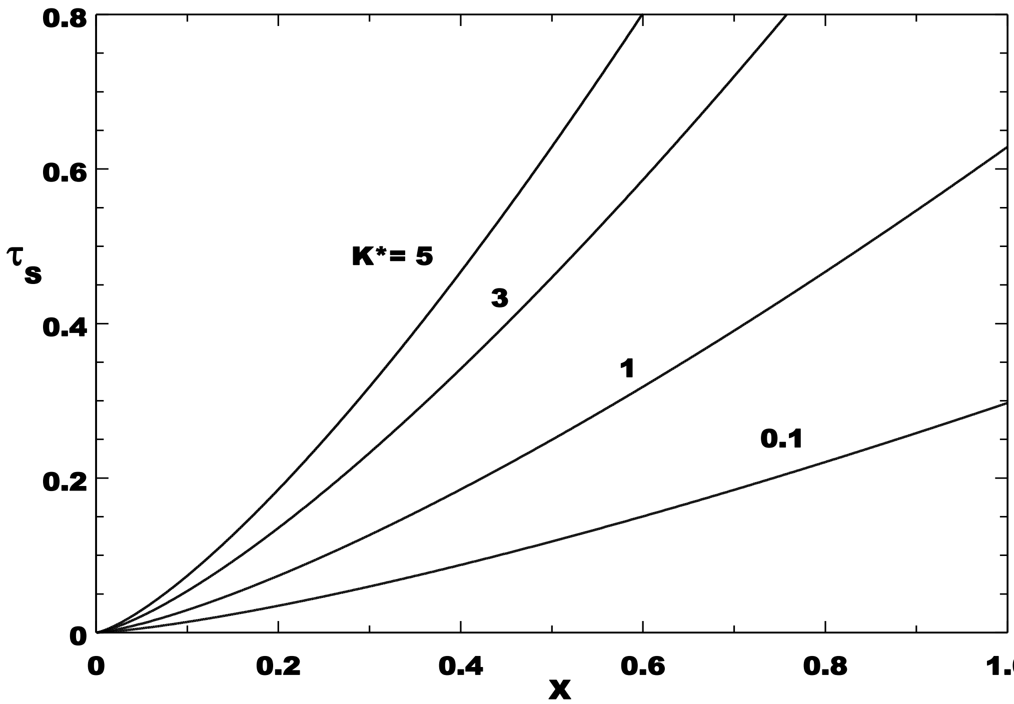

Figure 4: Effect of the permeability ratio K^* on the limiting line characteristic for $n = 1.2$, $\theta = 30°$ for the surface heated isothermally

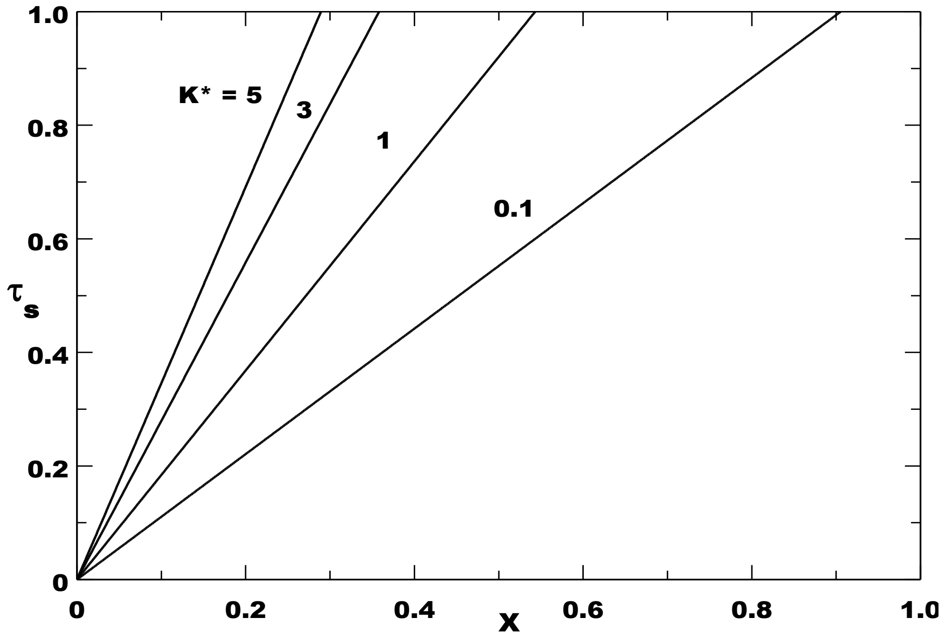

Figure 5: Effect of the permeability ratio K^* on the limiting line characteristic for $n = 1.2,\ \theta = 30°$ for the surface heated by a constant heat flux

In the same way, the same result is obtained when the wall is heated by a constant heat flux, a situation for which τ_s depends solely on $(K^*)^{0.45}$ and is proportional to it, as n, θ and X are made constant.

On the other hand, the same behavior is observed in Figures 6 and 7 illustrating the variation of τ_s versus X, for different values of θ, $n = 0.6$ and $K^* = 5$ and when the two types of thermal boundary conditions are considered here, respectively. Therefore, the limiting time to reach steady-state regime increases with an increase in orientation angle of the principal axes of the porous matrix, when other parameters are held constant.

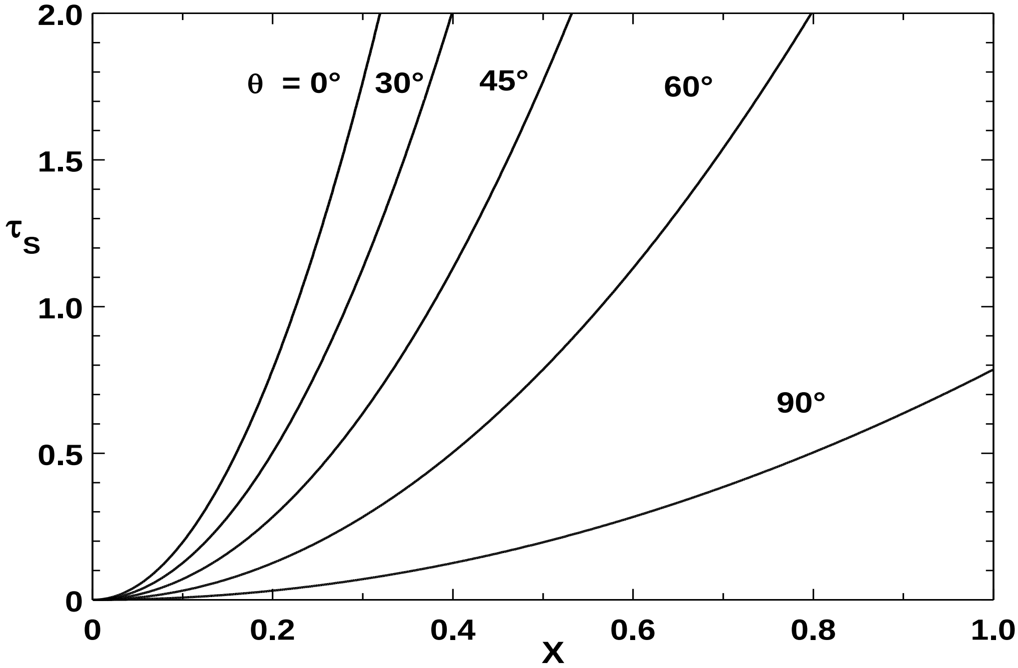

Figure 6: Effect of the orientation angle θ of the principal axes on the limiting line characteristic for $n = 0.6$, $K^* = 5$ for the surface heated isothermally

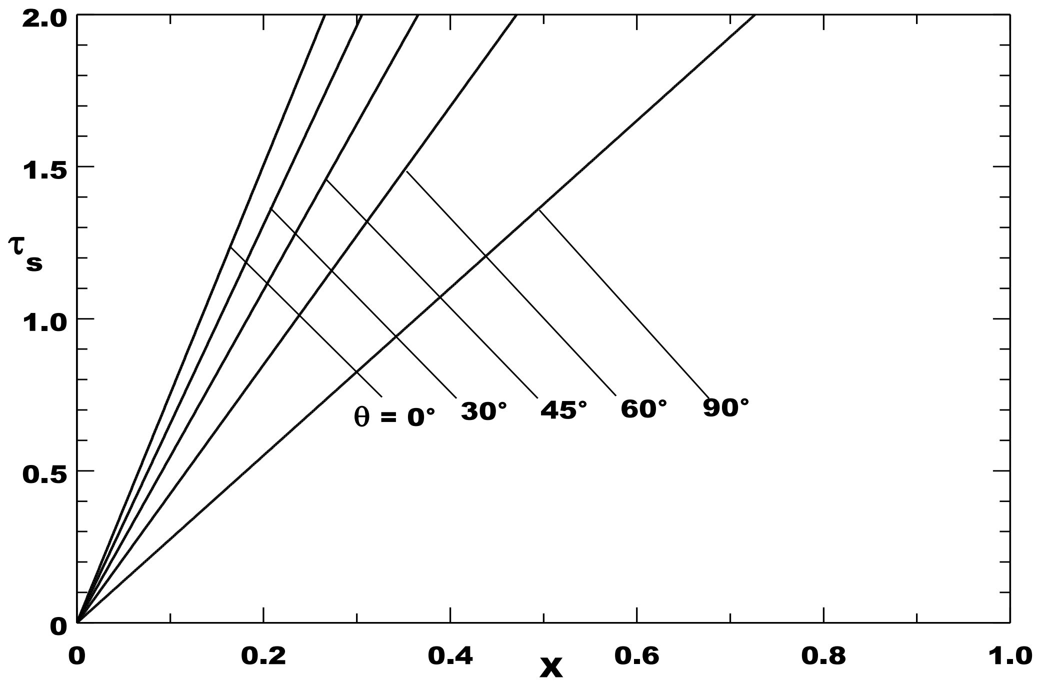

Figure 7: Effect of the orientation angle θ of the principal axes on the limiting line characteristic for $n = 0.6$, $K^* = 5$ for the surface heated by a constant heat flux.

Figures 8 and 9 show the effect of varying the modified Darcy-Rayleigh number and the time τ (lower than the limiting time required to reach steady-state τ_s) on the boundary-layer thickness A for $n = 0.6$ and for the two types of thermal boundary conditions considered here, respectively. As expected, the boundary-layer thickness A decreases drastically as Ra_L or R_L is made higher, giving rise to a channeling of convective heat flow near the horizontal surface.

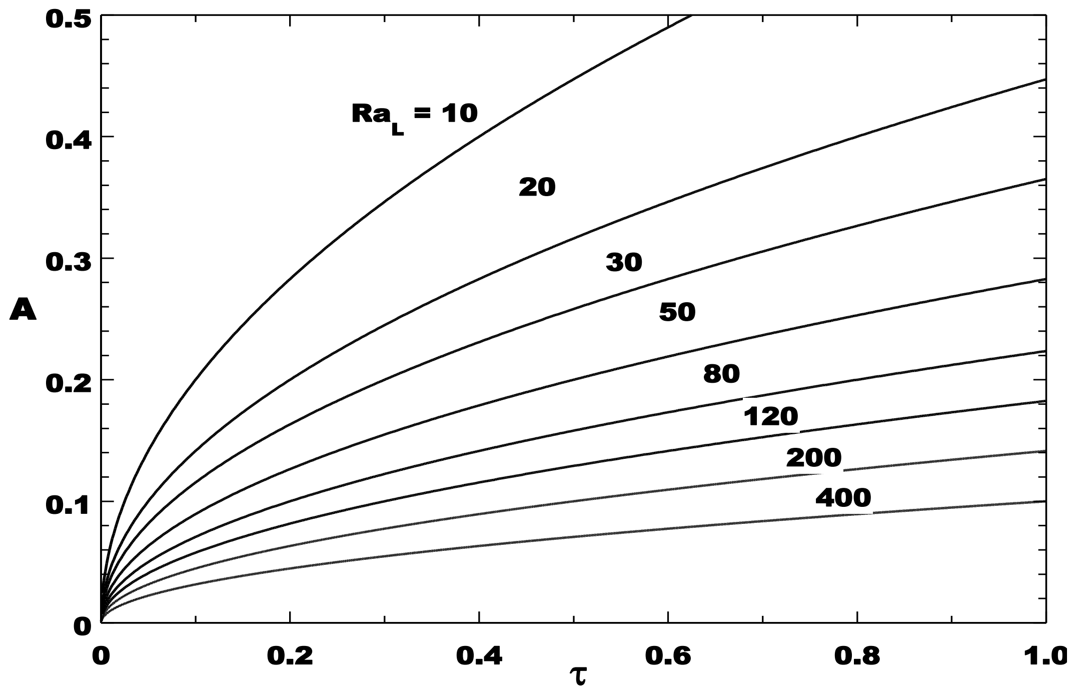

Figure 8: Effect of the time τ ($< \tau_s$) on the boundary-layer thickness A for $n = 0.6$ and various values of Rayleigh number for the surface heated isothermally

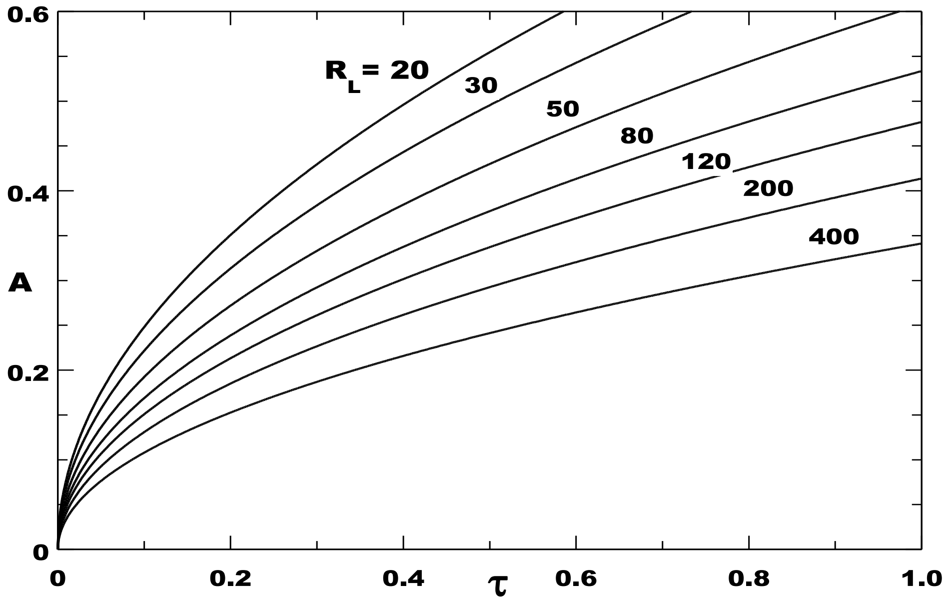

Figure 9: Effect of the time τ ($< \tau_s$) on the boundary-layer thickness A for $n = 0.6$ and various values of Rayleigh number for the surface heated by a constant heat flux

Moreover, it is seen that, for a given value of the time τ (lower than τ_s), the boundary-layer thickness A decreases with an increase in the modified Darcy-Rayleigh number. This trend follows from the fact that, according to equation (61), the boundary-layer thickness A is proportional to $\tau^{1/2}$ and inversely proportional to $Ra_L^{1/(2n+1)}$ such that, upon increasing Ra_L, A drops progressively and becomes less and less affected by τ. Similarly, for the heating process by a constant heat flux, according to equation (78), the boundary-layer thickness A is proportional to $\tau^{1/2}$ and inversely proportional to $R_L^{1/[2(n+1)]}$ such that, upon increasing R_L, A drops progressively and becomes less and less affected by τ.

Another view of the effects of varying of the power-law index n and the time ($\tau < \tau_s$) for $Ra_L = 30$ and for a step increase in wall temperature, is depicted in Figure 10 on the one hand, and for a step increase in wall constant heat flux in Figure 11 on the other hand.

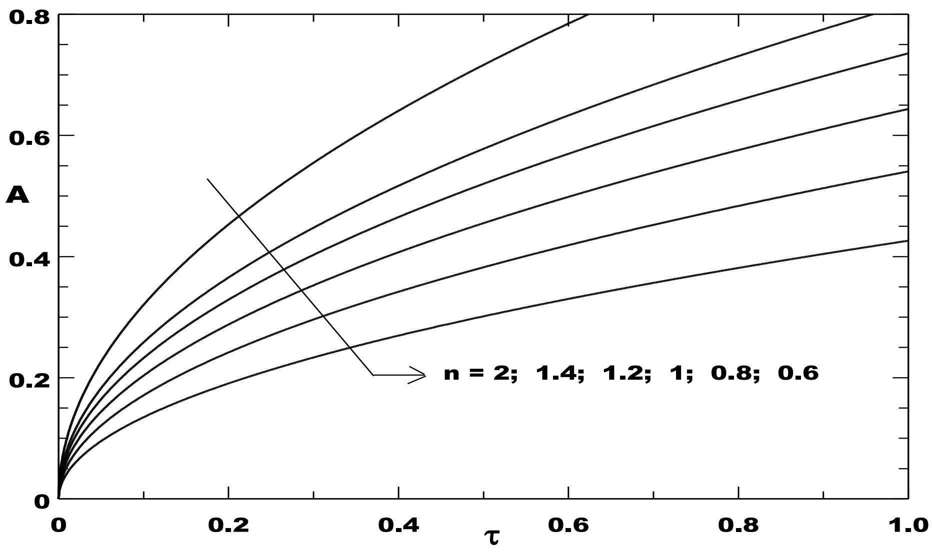

Figure 10: Effect of the time τ ($< \tau_s$) on the boundary-layer thickness A for $Ra_L = 30$ and for various values of power law index when the surface is heated isothermally

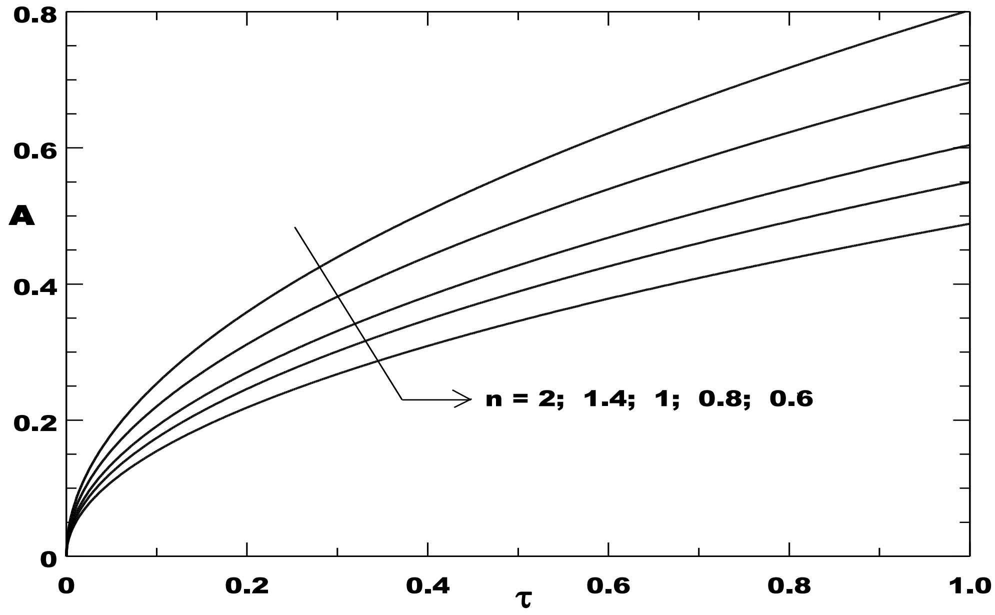

Figure 11: Effect of the time τ ($< \tau_s$) on the boundary-layer thickness A for $Ra_L = 30$ and for various values of power law index when the surface is heated by a constant heat flux

It is observed that the boundary-layer thickness A increases with an increase in power-law index n of non-Newtonian fluids. This can be explained that for a fixed value of τ, according to equation (61) when $n \to 0$, $Ra_L^{-1/(2n+1)} \to 0$, and therefore $A \to 0$. Similarly considering equation (78), when τ is held constant and $n \to 0$, $Ra_L^{-1/[2(n+1)]} \to 0$, and therefore $A \to 0$. So, in Figures 10 and 11, the boundary-layer thickness drops progressively as n is made weaker, and this independently of τ.

In Figures 12 and 13 the Nusselt Nu_x and the stream function Ψ for different position on the horizontal wall are given as a function of K^*, θ and n at steady-state. According to equations (70) and (71), it is observed that $Nu_x / Ra_x^{1/(2n+1)}$ and $\Psi/Ra_x^{1/(2n+1)}$ depend solely on n and $a^{-1/(2n+1)}$, where $a = \cos^2\theta + K^* \sin^2\theta$, i.e., on the anisotropic properties of the porous medium, namely, the permeability ratio K^* and the inclination angle θ.

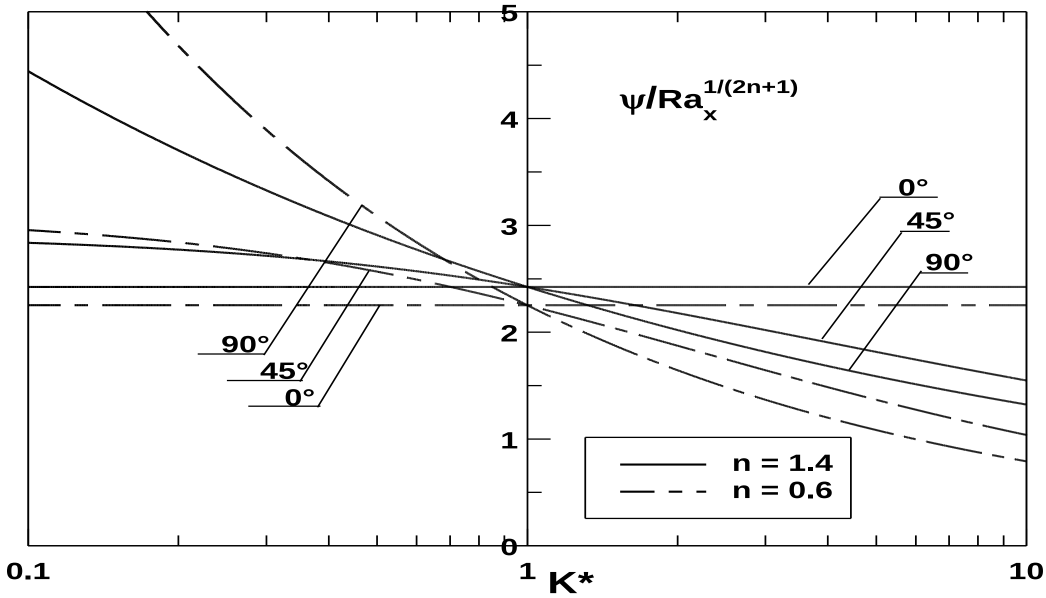

Figure 12: Effect of the permeability ratio K^* for various values of n and θ on ψ when the surface is heated isothermally

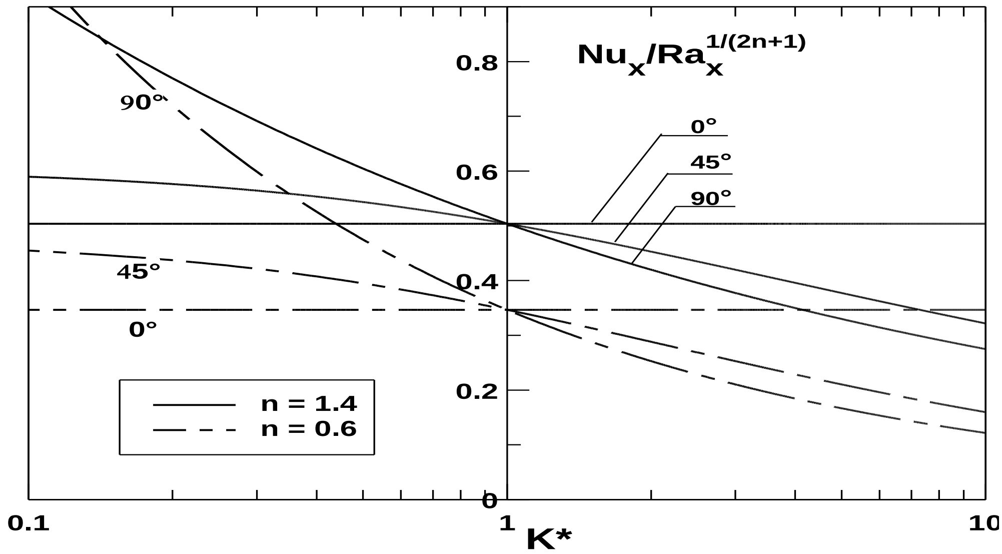

Figure 13: Effect of the permeability ratio K^* for various values of n and θ on Nu when the surface is heated isothermally

It is found that, for each value of n, when $K^* < 1$, the heat transfer rate and the stream function increase as θ decreases from 90° to 0°. However, it is seen that, when K^* >1, $Nu_x/Ra_x^{1/(2n+1)}$ and $\Psi/Ra_x^{1/(2n+1)}$ decrease as θ increases from 0° to 90°. That reversed situation comes from the fact the two terms $Nu_x/Ra_x^{1/(2n+1)}$ and $\Psi/Ra_x^{1/(2n+1)}$ are inversely proportional to a, thus to K^* and to θ. It is obvious that, by setting K^*= 1, the parameter a becomes equal to unity (corresponding to an isotropic porous medium) and consequently, both the heat transfer rate and the stream function are independent to anisotropic parameter.

6. Conclusions

Transient natural convection about a porous medium adjacent to a horizontal surface with a step increase in wall temperature or surface heat flux is analytically studied using the method of characteristics. The porous medium is anisotropic in permeability whose principal axes are non-coincident with the gravity vector and is saturated by a non-Newtonian fluid. The momentum equation is formulated on the basis of the modified Darcy power-law model of Pascal [20,21] and the generalized-Darcy's law proposed by Bear [22] . From the results, the following conclusions are drawn.

- The problem considered gives rise to the singularity problem in passing from the initial stage when the leading edge effect is not felt to the steady state defined for large times. The limiting line characteristic represents the time required to reach the steady state τ_s and the singularity value of time for which the heat transfer characteristics change suddenly from transient one-dimensional heat conduction to steady two-dimensional natural convection.
- For small values of time ($\tau < \tau_s$), the solutions for flow and temperature fields are dependent solely of time and the heat transfer due by pure conduction in transient regime are independent on the anisotropic parameters of the porous matrix and the power-law indexes of non-Newtonian fluids. Moreover, for a given value of the time τ (lower than τ_s), the boundary-layer thickness A decreases with an increase in the modified Darcy-Rayleigh number, such that, upon increasing Ra_L or R_L, A drops progressively and becomes less and affected by τ.
- For large values of time over the time required to reach the steady state ($\tau > \tau_s$), the solutions for stream function, temperature and heat transfer rate valid in steady regime are independent of time, and depend greatly on anisotropic parameters on the power-law indexes, and on the horizontal distance from the edge of the heated surface.
- At the steady regime the limiting line characteristic, the convection heat transfer is enhanced when the power-law index n of non-Newtonian fluids is increased.

- It has demonstrated that, for a given value of the distance X considered from the edge of the horizontal heated wall, the time taken by the heating process transfer to reach steady-state pattern for which convection occurs, increases with an increase in permeability ratio K^*, when other parameters are held constant. Similarly, the limiting time to reach steady-state regime increases with an increase in orientation angle θ of the principal axes of the porous matrix when other parameters are made fixed.

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