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Nonlocal Quantum Electrodynamics

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Abstract It is shown that an origin of the divergence problem in quantum electrodynamics is associated with a singularity of classical electrostatic field. A modification of its Coulomb potential at small distances leads to the change of the photon propagator which allows us to construct finite and gauge invariant quantum electrodynamics. We establish restriction on the value of the so - called fundamental length $l \lesssim 10^{-16}$ cm from the experimental data on the measuring anomalous magnetic moment of leptons. It is well known that any modification of the spinor propagator (in particular, electron one) gives rise to much problems connected with verification of basic principles of the theory like gauge invariance, unitarity, causality condition and so on. However, it turns out that square root modification of the spinor propagator is free from these difficult problems. Here we also construct a finite square root quantum electrodynamics.

Keywords quantum electrodynamics

1. Derivation of an Empirical Formula

A beautiful quantum electrodynamics developed by many physicists of the 20^{th} Century [1-5] has been played a vital role in the construction of the finite and gauge invariant so - called standard model [6-7] of the particle physics. What was an initial origin of this theory. It is natural that it was classical electrostatic field theory. In generally speaking, as usual, classical and quantum theories are the models of point - like particles. For example, the Newtonian and Coulomb potentials

$$U_N(r) = \frac{G}{4\pi r}, \qquad U_C(r) = \frac{e}{4\pi r} \tag{1}$$

are the potentials of the point - like sources of mass and charge, respectively:

$$\rho_N(r) = m\delta(r), \qquad \rho_C(r) = e\delta(r)$$

where $\delta(r) = \delta(x)\delta(y)\delta(z)$ is the Dirac δ - function with properties:

$$\int_{-\infty}^{\infty} dx\delta(x) = 1, \qquad \int_{-\infty}^{\infty} dx\delta(x)f(x) = f(0)$$

and etc.

It is well known that the inverse Fourier transform of the Coulomb potential for point - like charge is

$$D(p) = \frac{1}{e}\int d^3r e^{ipr} U_C(r) = \frac{1}{p^2} \tag{2}$$

and its relativistic generalization in four - momentum space

$$D(p) = \frac{1}{-p_0^2+p^2-i\varepsilon} \tag{3}$$

gives the local photon propagator which leads to the divergent theory. Fundamental importance is that the Coulomb potential (1) satisfies the Laplace equation

$$\Delta U_C(r) = 0 \tag{4}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In principle, any modification of the Coulomb potential at small distances leads to a violation of the Laplace equation (4). Here we find out more simple and natural changing of the Coulomb potential

$$U_C(r) \Rightarrow U_C^l(r) = \frac{e}{4\pi}\frac{1}{\sqrt{r^2+l^2}} \tag{5}$$

which does not satisfy the Laplace equation (4) and gives modification of the photon propagator (3):

$$D(p) \Rightarrow D^l(p) = \frac{1}{-p^2-i\varepsilon}V_l(-p^2l^2) \tag{6}$$

where

$$V_l(-p^2l^2) = \frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\eta\,\frac{v(\eta)}{\sin\pi\eta}[l^2(-p^2-i\varepsilon)]^{1+\eta} \tag{7}$$

$(1 < \beta < 2)$

$$v(\eta) = \frac{\pi}{4^{1+\eta}}\frac{1}{\sin\pi\eta\cdot\Gamma(1+\eta)\Gamma(2+\eta)} \tag{8}$$

Some time ago Markov [8] considered possibility of changing metric form

$$S_0 = x^2+y^2+z^2 \rightarrow x^2+y^2+z^2 \pm l^2$$

in his indefinite metric modification of the field theory.

The Poisson equation for the potential (5) takes the form

$$\Delta U = -\frac{3l^2}{(r^2+l^2)^{5/2}}$$

On the other hand the basic equation for electric stress $E = -\,grad\ \phi$ with extended charges is

$$div\ E = 4\pi\rho = -\,div\ \ grad\ \phi = -\Delta\phi,$$

$$\Delta\phi = -4\pi\rho$$

It means that in our case electric charge is not located at the single point and is distributed continuously over the whole space with the density

$$\rho = \frac{1}{4\pi}\frac{3l^2}{(r^2+l^2)^{5/2}}$$

with the normalization

$$\int d^3r\rho(r) = 1$$

as it should be.

Therefore, in our scheme, an idealized concept of the point - like charge is absent. Moreover, already in the early developments of quantum mechanics occur square - root operators. In particular, it was the relativistic relation between energy and momentum in a coordinate space representation that hindered its use [9]. A review of the early and later works are contained in [10]. In bound - state problems of two - and three - quark systems the Salpeter equation is often used [11-13]. Problems associated with binding in very strong fields [14-15], string theory [16-17] and astrophysical black holes [18-20] are applicable areas. Green's function for differential equations of infinite order like

$$\sqrt{m^2-\Box}\,\Omega(x) = -\delta^{(4)}(x) \tag{9}$$

are treated in [21]. Green function (9) in momenum p - and x - spaces take the forms

$$\Omega(p) = -\frac{1}{\sqrt{m^2-p^2-i\varepsilon}} = \int_{-m}^{m} d\lambda\rho_m(\lambda)\tilde{S}(\lambda,\hat{p}) \tag{10}$$

and

$$\Omega(x-y) = \int_{-m}^{m} d\lambda\rho_m(\lambda)S(x-y,\lambda) \tag{11}$$

where the distribution

$$\rho_m(\lambda) = \frac{1}{\pi}(m^2-\lambda^2)^{-1/2}$$

has properties, like

$$\int_{-m}^{m} d\lambda\rho_m(\lambda) = 1, \qquad \int_{-m}^{m} d\lambda\cdot\lambda\cdot\rho_m(\lambda) = 0$$

$$\int_{-m}^{m} d\lambda\lambda^2\rho_m(\lambda) = \frac{1}{2}m^2 \tag{12}$$

and

$$\tilde{S}(\lambda,\hat{p}) = \frac{1}{i}\frac{\lambda+\hat{p}}{\lambda^2-p^2-i\varepsilon} \tag{13}$$

$$S(x-y,\lambda)=\frac{1}{(2\pi)^4}\frac{1}{i}\int d^4pe^{-ip(x-y)}\frac{\lambda+\hat{p}}{\lambda^2-p^2-i\varepsilon} \tag{14}$$

are the Dirac spinor propagator in corresponding spaces with random mass λ. Here the relations

$$m^2-p^2=(m-\hat{p})(m+\hat{p}),\qquad \hat{p}=\gamma^\nu p_\nu$$

and the Feynman parametric formula

$$\frac{1}{a^{n_1}b^{n_2}}=\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1 dxx^{n_1-1}(1-x)^{n_2-1}\frac{1}{[ax+b(1-x)]^{n_1+n_2}} \tag{15}$$

are used. In this paper by using formulas (6), (10), (11), (13) and (14) we will construct finite nonlocal and square - root quantum electrodynamics free from ultraviolet divergences.

2 Modification of the Coulomb Potential and Derivation of the Nonlocal Photon Propagator

We propose the following finite Coulomb potential at small distances:

$$U_C^l(r)=\frac{e}{4\pi}\frac{1}{\sqrt{x^2+y^2+z^2+l^2}} \tag{16}$$

where l - is some parameter dimension of length. Its value may be interpreted as a size of an extended electric charge or as an universal constant like fundamental length in physics. As mentioned above this modified potential satisfies the Poisson equation. Let us calculate the Fourier transform of the finite potential (16):

$$D_l(p)=\frac{1}{e}\int d^3re^{ipr}\left(\frac{e}{4\pi\sqrt{r^2+l^2}}\right)=\frac{1}{p}\int_0^\infty dr\frac{r}{\sqrt{r^2+l^2}}\sin pr,$$

$(p=|p|)$. By using the Mellin representation this expression takes the form

$$D_l(p)=\frac{l^2}{2\sqrt{\pi}}\cdot\frac{l}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty}d\eta\frac{(p^2l^2)^\eta}{\sin\pi\eta\,\Gamma(2+2\eta)}\times\Gamma\left(\frac{3}{2}+\eta\right)\Gamma(-1-\eta) \tag{17}$$

where $(1<\beta<2)$.

Further, taking into account Gamma - function relations:

$$\Gamma(2+2\eta)=\frac{2^{2(1+\eta)-1}}{\sqrt{\pi}}\Gamma(1+\eta)\Gamma\left(\frac{3}{2}+\eta\right)$$

$$\Gamma(\eta)\Gamma(1-\eta)=\frac{\pi}{\sin\pi\eta}$$

and after some elementary calculations, one gets

$$D_l(p)=\frac{V_l(p^2l^2)}{p^2} \tag{18}$$

where

$$V_l(p^2l^2)=\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty}d\eta\frac{v(\eta)}{\sin\pi\eta}[l^2p^2]^{1+\eta} \tag{19}$$

$$v(\eta)=\frac{\pi}{4^{1+\eta}}\frac{1}{\sin\pi\eta}\frac{1}{\Gamma(1+\eta)\Gamma(2+\eta)} \tag{20}$$

From these formulas one can calculate residues at the points $\eta=-1,0,1,\ldots$. The result reads

$$D_l(p)=\frac{l}{|p|}K_1(l|p|) \tag{21}$$

where $K_1(x)$ is the modified Bessel function of second kind or the Mac'Donald function

$$K_1(x)=\frac{\pi}{2}\frac{x}{2}\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty}d\zeta\frac{\left(\frac{x}{2}\right)^{2\zeta}}{\sin^2\pi\zeta\,\Gamma(1+\zeta)\Gamma(2+\zeta)}$$

$(0<\beta<1)$, $x=|p|l$.

Finally, the modification of the Coulomb potential (16) gives rise to the following nonlocal photon propagator [22]

$$D_{\mu\nu}^l(x)=\frac{i}{(2\pi)^4}g_{\mu\nu}\int d^4pe^{ipx}\frac{V_l(-p^2l^2)}{-p^2-i\varepsilon} \tag{22}$$

where the form - factor $V_l(-p^2l^2)$ of the theory is defined by the formulas (19) and (20).

Here our theory with the propagator (22) is very similar to the nonlocal theory due to [22] and [23]. Notice that the simple modification of the Coulomb potential (16) leading to the nonlocal photon propagator (22) is cornerstone of the finiteness of classical and quantum electromagnetic fields. For example, now electrostatic self - energy of the extended charge is finite at small distances:

$$W=\frac{e}{2}\int d^3r\rho(r)U_l(r)=\frac{1}{2}\int d^3rE^2,\quad E=-\frac{e}{4\pi}\,grad\,\frac{1}{\sqrt{r^2+l^2}}$$

Here simple calculation reads

$$W = \frac{e^2}{8\pi}\int_0^\infty dr \frac{r^4}{[r^2+l^2]^3} = \frac{e^2}{2\Gamma(3)}\frac{1}{l}\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{8\pi} = \frac{3}{32}\frac{\pi}{l}\alpha$$

Moreover, the nonlocal photon propagator (22) is finite at the origin

$$D^l_{\mu\nu}(0) = g_{\mu\nu}\frac{2\pi^2}{2^4\pi^4}\int_0^\infty dpp^3 D_l(p^2) =$$

$$= -g_{\mu\nu}\frac{1}{8\pi}\lim_{\epsilon\to 0}\int_{-\beta+i\infty}^{-\beta-i\infty} d\zeta \frac{\left(\frac{l^2}{4}\right)^{1+\zeta}}{\sin^2\pi\zeta}\frac{\varepsilon^{2\zeta+4}}{\Gamma(1+\zeta)\Gamma(2+\zeta)\Gamma(2\zeta+4)}$$

where $2 < \beta < 3$.

Calculation of residue at the point $\zeta = -2$ and taking the limit $\varepsilon \to 0$ leads

$$D^l_{\mu\nu}(0) = \frac{1}{4\pi^2 l^2} g_{\mu\nu} = const$$

It turns out that, in principle, due to finiteness of $D^l_{\mu\nu}(0)$ one can calculate vacuum fluctuation diagrams, shown in Figure 1.

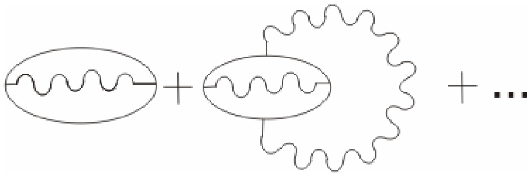

Figure 1: Primative Feynman diagrams for vacuum fluctuation

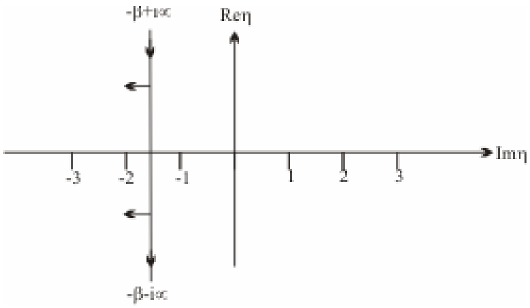

Figure 2: Integration contour in the formula (19)

Finally, we indicate one important consequence of the photon propagator (22) with the form - factor (19). If we want to calculate high order divergence integrals over the internal momentum variable p, like

$$\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(\eta)}{\sin\pi\eta}\int d^4p \frac{[p^{2\nu}]^\eta}{[p^2+A]^\lambda}$$

for any order of ν, then we can move integration contour in Figure 2 to the left through points $\eta = -2, -3, \ldots$, in desired order, since in such type of integrals there are no poles at these points. After integration result we can again move integration contour to the right to calculate residues at the points $\eta = -3, -2, -1, \ldots$ so on. This procedure of analytic continuation over complex variable η plays a vital role in regularization scheme.

3. Nonlocal Quantum Electrodynamics

3.1. Introduction

Lagrangian functions of the nonlocal quantum electrodynamics arisen from the modification of the Coulomb potential at small distances have similar structures as the local theory [24].

$$\begin{aligned} L(x) &= e{:}\overline{\psi}(x)\hat{A}(l,x)\psi(x){:} \\ &+e(Z_1-1){:}\overline{\psi}(x)\hat{A}(l,x)\psi(x){:} \\ &-\delta m{:}\overline{\psi}(x)\psi(x){:} + (Z_2-1){:}\overline{\psi}(x)(i\hat{\partial}-m)\psi(x){:} \\ &-(Z_3-1)\frac{1}{4}{:}F_{\mu\nu}(x)F^{\mu\nu}(x){:} \end{aligned} \tag{23}$$

where

$$\hat{A}(l,x) = A_\mu(l,x)\gamma^\mu, \quad \hat{\partial} = \gamma^\mu\frac{\partial}{\partial x_\mu}$$

Only in our case of the nonlocal theory, renormalization constants $Z_1, Z_2, Z_3, \delta m$ are finite and moreover $Z_1 = Z_2$ due to the Ward - Takahashi identity. Here "chronological" pairing (or T - product) of the fermonic field operators of electrons has the usual local form:

$$S(x-y) = \langle 0|T[\psi(x)\overline{\psi}(y)]|0\rangle = \frac{1}{(2\pi)^4}\frac{1}{i}\int d^4p \frac{e^{-ip(x-y)}}{m-\hat{p}-i\varepsilon} \tag{24}$$

while "causal" function of the nonlocal electromagnetic field $A_\mu(l,x)$ in (23) takes the form due to the formula (22)

$$D^l_{\mu\nu}(x-y) = g_{\mu\nu} D^l(x-y) = -\frac{g_{\mu\nu}}{(2\pi)^4 i}\int d^4p e^{-ip(x-y)}\frac{V_l(-p^2l^2)}{-p^2-i\varepsilon} \tag{25}$$

where $V_l(-p^2l^2)$ is given by formulas (19) and (20).

3.2. The Electron Self - Energy in NQED

The complete electron propagator in NQED is given by the sum

$$[-i(2\pi)^{-4}S_l'(p)] = [-i(2\pi)^{-4}S(p)] + [i(2\pi)^{-4}S(p)][i(2\pi)^4\Sigma_l(p)]$$
$$\times[-i(2\pi)^{-4}S(p)] + \cdots$$

where

$$S(p) = \frac{m+\hat{p}}{m^2-p^2-i\varepsilon}$$

The sum is trivial and gives

$$S_l'(p) = [m-\hat{p}-\Sigma_l-i\varepsilon]^{-1}$$

In lowest order there is a one - loop contribution to Σ_l, given by in Figure 3:

$$-i{:}\overline{\psi}(x)\Sigma_l(x-y)\psi(y){:}$$

where

$$\Sigma_l(x-y) = -ie^2\gamma_\mu S(x-y)\gamma_\mu D^l(x-y) \tag{26}$$

Passing to the momentum representation and going to the Euclidean metric by using $k_0 \to \exp(i\pi/2)k_4$, one gets

$$\widetilde{\Sigma}_l(p) = \frac{e^2}{(2\pi)^4}\int d^4k_E \frac{V_l(k_E^2l^2)}{k_E^2}\gamma_\mu^{(E)}\frac{m-\hat{p}_E+\hat{k}_E}{m^2+(p_E-k_E)^2}\gamma\mu^{(E)}$$

Here $p_E = (-ip_0, p)$, $\gamma^{(E)} = (-i\gamma_0, \vec{\gamma})$ and $k_E = (k_4, k)$. Taking into account the Mellin representation (19) for the form - factor $V_l(k_E^2l^2)$ and after some calculations, we have

$$\widetilde{\Sigma}_l(p) = -\frac{e^2}{8\pi}\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{1}{\sin^2\pi\eta}\frac{v(\eta)(m^2e^2)^{1+\eta}}{\Gamma(2+\eta)}F(\eta,p) \tag{27}$$

where

$$F(\eta,p) = \frac{1}{\Gamma(-\eta)}\int_0^1 du\left(\frac{1-u}{u}\right)^{1+\eta}\left(1-\frac{p^2}{m^2}u\right)^{1+\eta}(2m-\hat{p}u) \tag{28}$$

is a regular function in the half - plane $Re\,\eta > -2$.

Assuming the value m^2l^2 to be small, one can obtain (after calculation of residues at the points $\eta = -1{,}0$):

$$\widetilde{\Sigma}_l(p) = \frac{e^2}{8\pi^2}\int_0^1 du(2m-\hat{p}u)\ln\left(1-\frac{p^2}{m^2}u\right) +$$
$$+\frac{e^2}{8\pi^2}\left\{\ln\left(\frac{m^2l^2}{4}\right)\left(2m-\frac{1}{2}\hat{p}\right)+\hat{p}\left(\frac{1}{2}+\psi(1)\right)-4m\psi(1)\right\}+$$
$$+\frac{me^2}{32\pi^2}(m^2l^2)\left[\ln^2\left(\frac{m^2l^2}{4}\right)-\ln\left(\frac{m^2l^2}{4}\right)(3+4\psi(1))+\right.$$
$$\left.+4\psi(1)(1+\psi(1))+2-\frac{1}{3}\pi^2\right], \tag{29}$$

where $\psi(1) = -C, C = 0.57721566490$ ... is the Euler number.

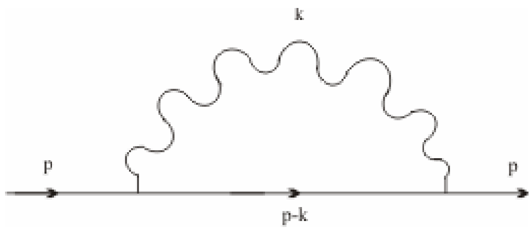

Figure 3: Diagram of Self - energy of a electron in NQED

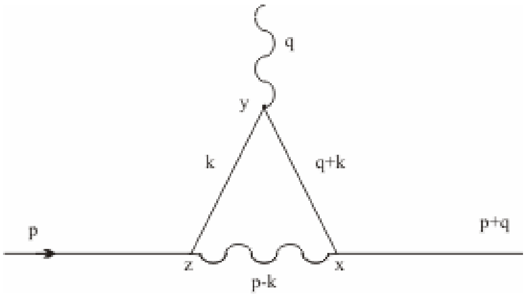

Figure 4: Vertex function in NQED

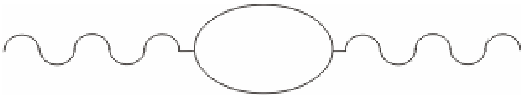

Figure 5: The vacuum polarization in NQED

3.3 Vertex Function and Anomalous Magnetic Moment of Leptons in NQED

Let us consider Feynman diagram shown in Figure 4. The following matrix element corresponds to this diagram:

$$ie : \overline{\psi}(x)\Gamma^l_\mu(x, z/y)\psi(z)A_\mu(y): \tag{30}$$

Analogously, in the momentum space and in the Euclidean metric, the vertex function takes the form

$$\widetilde{\Gamma}^l_\mu(p_1, p) = -\frac{e^2}{(2\pi)^4}\int d^4k_E \frac{V_l((p_E-k_E)^2 l^2)}{(p_E-k_E)^2}\gamma_\nu \times$$
$$\times \frac{m-\hat{k}_E-\hat{q}_E}{m^2+(k_E+p_E)^2}\gamma_\mu \frac{m-\hat{k}_E}{m^2+k_E^2}\gamma_\nu \tag{31}$$

Again passing to the Minkowski metric and using the generalized Feynman parameterization formula (15), one gets

$$\widetilde{\Gamma}^l_\mu(p_1; p) = \frac{e^2}{8\pi}\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(n)}{\sin^2\pi\eta}\frac{(m^2l^2)^{1+\eta}}{\Gamma(2+\eta)} F\mu(\eta; p_1, p) \tag{32}$$

where

$$F_\mu(\eta; p_1, p) = \gamma_\mu F_1(\eta; p_1, p) + F_2(\eta; p_1, p)$$

Here

$$F_1(\eta; p_1, p) = \frac{1}{\Gamma(-\eta)}\int_0^1\int_0^1\int_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma)\alpha^{-1-\eta}Q^{1+\eta}$$
$$F_2(\eta; p_1, p) = \frac{1}{\Gamma(-1-\eta)}\int_0^1\int_0^1\int_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma)\alpha^{-1-\eta}Q^{\eta} \times$$
$$\times \frac{1}{m^2}\left[m^2\gamma_\mu - 2mq_\mu + 4m(\beta q_\mu - \alpha p_\mu) +\right.$$
$$\left. +(\alpha\hat{p} - \beta\hat{q})\gamma_\mu\hat{q} + (\alpha\hat{p} - \beta\hat{q})\gamma\mu(\alpha\hat{p} - \beta\hat{q})\right] \tag{33}$$

$$Q = \beta + \gamma - \alpha\gamma\frac{p^2}{m^2} - \beta\gamma\frac{q^2}{m^2} - \alpha\beta\frac{(p+q)^2}{m^2} \tag{34}$$

Let us calculate the vertex function (32) for two cases: first, when $q = 0$ and p has an arbitrary value; second, when q is an arbitrary quantity and p, p_1 are situated on the m - mass shell. In the first case, assuming $q = 0$ in the formula (33) and after some standard calculations, one gets

$$F_\mu(\eta; p_1, p) = \frac{1}{\Gamma(-\eta)}\int_0^1 du \left(\frac{1-u}{u}\right)^{1+\eta}\left(1 - u\frac{p^2}{m^2}\right)^{1+\eta} \times$$
$$\times \left[u\gamma_\mu + \frac{2(1+\eta)up_\mu(2m-u\hat{p})}{m^2-up^2}\right] \tag{35}$$

Comparing this formula with the expression (28) for the self - energy of the electron, it is easily seen that

$$F_\mu(\eta; p_1, p) = -\frac{\partial}{\partial p_\mu}F(\eta; p) \tag{36}$$

From this identity, we can obtain a very important conclusion. In nonlocal QED constructed using the modification of the Coulomb potential, the Ward - Takahashi identity is valid:

$$\widetilde{\Gamma}^l_\mu(p, p) = -\frac{\partial}{\partial p_\mu}\widetilde{\Sigma}_l(p) \tag{37}$$

In the second case, one can put

$$\overline{u}(p_1)\tilde{\Gamma}^l_\mu(p_1,p)u(p) = \overline{u}(p_1)\Lambda_\mu(q)u(p) \tag{38}$$

where $u(p_1)$ and $u(p)$ are solutions of the Dirac equation

$$(\hat{p}-m)u(p)=0, \quad \overline{u}(p_1)(\hat{p}_1+m)=0$$

Substituting the vertex function (32) into (38) and after some transformations, we have

$$\overline{u}(p_1)F_\mu(\eta;p_1,p)u(p) = u(p_1)\Lambda_\mu(\eta;q)u(p) \tag{39}$$

Here

$$\Lambda_\mu(\eta;q) = \gamma_\mu f_1(\eta;q^2) + \frac{i}{2m}\sigma_{\mu\nu}q_\nu f_2(\eta;q^2)$$

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

$$f_j(\eta;q^2) = \frac{1}{\Gamma(-\eta)}\int_0^1\int_0^1\int_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma)\alpha^{-1-\eta}L^\eta \times g_j(\alpha,\beta,\gamma,q^2), \quad (j=1,2)$$

$$L = \varepsilon\alpha + (1-\alpha)^2 - \beta\gamma\frac{q^2}{m^2} \tag{40}$$

$$g_1(\alpha,\beta,\gamma,q^2) = (1-\alpha)^2(-\eta) + 2\alpha(1+\eta) -$$

$$-\frac{q^2}{m^2}[\beta\gamma + (1+\eta)(\alpha+\beta)(\alpha+\gamma)]\cdot$$

$$\cdot\, g_2(\alpha,\beta,\gamma,q^2) = 2\alpha(1-\alpha)(1+\eta)$$

To avoid infrared divergences in the vertex function we have introduced here the parameter $\varepsilon = m_{ph}^2/m^2$, taking into account the "mass" of the photon. Finally, one gets

$$\Lambda_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q_\nu F_2(q^2) \tag{41}$$

where

$$F_j(q^2) = \frac{e^2}{8\pi}\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\eta \frac{v(\eta)}{\sin^2\pi\eta}\frac{(m^2l^2)^{1+\eta}}{\Gamma(2+\eta)} f_j(\eta;q^2) \tag{42}$$

It is easy to verify that the vertex function $\Lambda_\mu(q)$ satisfies the gauge invariant condition:

$$q_\mu\overline{u}(p_1)\Lambda_\mu(q)u(p) = 0 \tag{43}$$

Let us write the first terms of the decomposition for the functions $F_1(q^2)$ and $F_2(q^2)$ over small parameters m^2l^2 and q^2/m^2:

$$F_2(q^2) = -\frac{\alpha}{2\pi}\left[1 + \frac{m^2l^2}{6}\left(\ln\frac{m^2l^2}{4} - 2\psi(1) + \frac{1}{6}\right)\right] + O\left(\frac{q^2}{m^2}\right) \tag{44}$$

$$F_1(q^2) = -\frac{\alpha}{4\pi}\left\{3\left[\ln\frac{m^2l^2}{4} - 2\psi(1) - \frac{3}{2}\right] + \right.$$

$$\left. + m^2l^2\left[\ln\frac{m^2l^2}{4} - 2\psi(1) - \frac{1}{3}\right]\right\} + O\left(\frac{q^2}{m^2}\right) \tag{45}$$

From this first formula we can see that corrections to the anomalous magnetic moment (AMM) for leptons are given by

$$\Delta\mu = \frac{\alpha}{2\pi}\left[1 + \frac{m^2l^2}{6}\left(\ln\frac{m^2l^2}{4} + \frac{1}{6} - 2\psi(1)\right)\right] \tag{46}$$

We seen that the first term in (46) is exactly famous Schwinger correction obtained in local QED. From the experimental values of the AMM of the electron and muon [25-27] and [28]

$$\Delta\mu_{exp}^{(e)} = \frac{\mu_e}{\mu_B} - 1 = \frac{1}{2}(q-2) = (1159652180.73(0.28)) \times 10^{-12} \tag{47}$$

and

$$\Delta\mu_{exp}^{(e)} = \frac{\mu_\mu}{(e\hbar/2m_\mu)} - 1 = \frac{1}{2}(g_\mu - 2) = (116592089(63)) \times 10^{-11} \tag{48}$$

one gets the following restriction on the value of the universal parameter (or the fundamental length) l:

$$l \lesssim 7.0 \times 10^{-17}\ cm\ for \quad \Delta\mu_{\exp}^{(e)} \tag{49}$$

$$l \lesssim 2.6 \times 10^{-17}\ cm\ for \quad \Delta\mu_{exp}^{(\mu)} \tag{50}$$

Recent theoretical calculations of the AMM of the electron and muon have been carried by [29].

3.4. Vacuum Polarization

Since, in our scheme the propagator $S(x-y)$ of the charged lepton spinor is not changed, the diagrams of the vacuum polarization i.e., closed spinor propagators (see Figure 5) of the leptons in our nonlocal QED are studied by the same way as in the local theory. For completeness we calculate it in e^2-order by using d -dimensional regularization procedure [30]. The result reads in the momentum space:

$$\widetilde{\prod}^{\rho\sigma}(q) = (q^2 g^{\rho\sigma} - q^\rho q^\sigma)\widetilde{\prod}(q^2) \tag{51}$$

where

$$\widetilde{\prod}(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx(1-x)\ln\left(1+\frac{q^2 x(1-x)}{m^2}\right) \tag{52}$$

The physical importance of the vacuum polarization in NQED can be explored by considering its effects on the scattering of two charged particles of spin $1/2$.

4. The Square - Root Nonlocal Quantum Electrodynamics

The purpose of this section is to study nonlocal interactions of the charged square - root spinors with nonlocal photons within our scheme. Thus, the Lagrangian corresponding to the equation

$$\sqrt{m^2-\Box}\phi(x) = 0 \tag{53}$$

is given by

$$L_\phi^0 = \phi^*(x)\sqrt{m^2-\Box}\phi(x) \tag{54}$$

Instead of (54) we consider the Lagrangian density

$$L_\psi^0 = -N\{\overline{\psi}(x,\lambda_1)(-\hat{\partial})\psi(x,\lambda_2) + L_{1\psi}^0\} \tag{55}$$

for the $\psi(x,\lambda)$ field. Here notation is used

$$L_{1\psi}^0 = \Psi(x,\lambda_1)U(\lambda_1,\lambda_2)\Psi(x,\lambda_2)$$

$$N = \int_{-m}^m\int_{-m}^m d\lambda_1 d\lambda_2 \rho(\lambda_1)\rho(\lambda_2), \quad \hat{\partial} = i\gamma^\mu\frac{\partial}{\partial x_\mu} \tag{56}$$

$$\overline{\Psi}_{(}x,\lambda_1) = (0,\overline{\psi}(x,\lambda_1))$$

$$\Psi(x,\lambda_2) = \begin{pmatrix}\psi(x,\lambda_2)\\ 0\end{pmatrix}$$

$$U(\lambda_1,\lambda_2) = \begin{pmatrix}0 & \lambda_1\\ \lambda_2 & 0\end{pmatrix}$$

Equations of motion

$$\int_{-m}^m d\lambda\rho(\lambda)(\hat{\partial}-\lambda)\psi(x,\lambda) = 0$$

$$\int_{-m}^m d\lambda\rho(\lambda)\left(i\frac{\partial\overline{\psi}(x,\lambda)}{\partial x_\mu}\gamma^\mu + \lambda\overline{\psi}(x,\lambda)\right) = 0 \tag{57}$$

for $\psi(x,\lambda)$ fields can be obtained from the action

$$A = \int d^4x L_\psi^0(x)$$

by using independent variations over the fields $\psi(y,\lambda)$ and $\overline{\psi}(y,\lambda)$ and by taking the differentiation $\delta L_{1\psi}^0/\delta\overline{\psi}(y,\lambda)$ and $\delta(L_{1\psi}^0)^T/\delta\psi(y,\lambda)$. Here we have used the following obvious relations

$$\frac{\delta\overline{\psi}(x,\lambda_i)}{\delta\overline{\psi}(y,\lambda)} = \frac{\delta\psi(x,\lambda_i)}{\delta\psi(y,\lambda)} = \delta^{(4)}(x-y)\delta(\lambda_i-\lambda)$$

and definition

$$(L_{1\psi}^0)^T = \overline{\Psi}(x,\lambda_1)U^T(\lambda_1,\lambda_2)\Psi(x,\lambda_2)$$

It is easily seen that the propagator of the field $\phi(x)$ in (53) is given by equation (9) or

$$\Omega(x) = -\frac{1}{\sqrt{m^2-\Box}}\delta^{(4)}(x) = \frac{1}{i}\int_{-m}^m d\lambda\rho(\lambda)\frac{1}{\lambda+\hat{\partial}}\delta^{(4)}(x) = \int_{-m}^m d\lambda\rho(\lambda)S(x,\lambda) \tag{58}$$

In the momentum representation, expression (58) takes the form

$$\widetilde{\Omega}(p) = \int_{-m}^{m} d\lambda\rho(\lambda)\tilde{S}(\lambda,\hat{p}) \quad (59)$$

where

$$\tilde{S}(\lambda,\hat{p}) = \frac{1}{i}\frac{\lambda+\hat{p}}{\lambda^2-p^2-i\varepsilon} \quad (60)$$

is the spinor propagator with random "mass" λ in momentum space.

Our next goal is to study Feynman diagrams in nonlocal square - root quantum electrodynamics with Green functions (22), (58), (59), (60).

In the "square - root" NQED the S - matrix can be constructed by the usual rule:

$$S = ExpecT\exp[\int d^4xL_{in}(x)] \quad (61)$$

where

$$L_{in}(x) = eN\{\overline{\psi}(x,\lambda_1)\hat{A}_l(x)\psi(x,\lambda_2)\} \quad (62)$$

$$\hat{A}_l = \gamma^\mu A_\mu^l(x)$$

and N is given by (56). The symbol T is defined by

$$\langle 0|T[\psi(x,\lambda_1)\overline{\psi}(y,\lambda_2)]|0\rangle = \delta(\lambda_1-\lambda_2)S(x-y,\lambda_1)/\rho(\lambda_1) \quad (63)$$

for the spinor fields. For example, at least for connected diagrams in the momentum space one assumes

$$Expec\{\widetilde{\Omega}(p)\} = \int_{-m}^{m} d\lambda\rho(\lambda)\tilde{S}(\hat{p},\lambda)$$

$$Expec\{\gamma^{\nu_1}\widetilde{\Omega}(p_1)\gamma^{\nu_2}\widetilde{\Omega}(p_2)\gamma^{\nu_3}\} =$$

$$= \int_{-m}^{m} d\lambda\rho(\lambda)\{\gamma^{\nu_1}\tilde{S}(\hat{p}_1,\lambda)\gamma^{\nu_2}\tilde{S}(\hat{p}_2,\lambda)\gamma^{\nu_3}\} \quad (64)$$

and so on.

The gauge invariance of the "square - root" NQED means that every matrix elements of the S - matrix (61) defining the concrete electromagnetic processes have a definite structure, and algebraical relations exist between them. In particular, in the momentum representation, the so - called vacuum polarization diagram like (in Figure 4) in the second order of the perturbation theory has the form

$$\widetilde{\prod}_{\mu\nu}^{l,s}(k) = (k_\mu k_\nu - g_{\mu\nu}k^2)\widetilde{\prod}^{l,s}(k^2) \quad (65)$$

and the relation

$$\frac{\partial\tilde{\Sigma}_l^s(p)}{\partial p_\mu} = -\tilde{\Gamma}_\mu^{l,s}(p,q)|_{q=0} \quad (66)$$

is valid between the vertex function $\tilde{\Gamma}_\mu^{l,s}(p,q)$ and the self - energy of the "square - root" electron $\tilde{\Sigma}_l^s(p)$. The relation (66) generalizes the Ward - Takahashi identity in QED. Here in accordance with (64), we have

$$\tilde{\Sigma}_l^s(p) = \frac{-ie^2}{(2\pi)^4}\int_{-m}^{m} d\lambda\rho(\lambda)\int d^4kD_l(k^2)\gamma^\mu S(\hat{p}-\hat{k},\lambda)\gamma^\mu \quad (67)$$

and

$$\tilde{\Gamma}_\mu^{l,s}(p,q) = \frac{ie^2}{(2\pi)^4}\int d^4kD_l((p-k)^2l^2)\times$$

$$\times E_{xpec}\{\gamma^\nu\widetilde{\Omega}(q+k)\gamma^\mu\widetilde{\Omega}(k)\gamma^\nu\} =$$

$$= \frac{ie^2}{(2\pi)^4}\int_{-m}^{m} d\lambda\rho(\lambda)\int d^4kD_l((p-k)^2l^2)\gamma^\nu S(\hat{q}+\hat{k},\lambda)\gamma^\mu\times$$

$$\times\tilde{S}(\hat{k},\lambda)\gamma^\nu \quad (68)$$

where

$$\tilde{S}(\hat{p},\lambda) = \frac{1}{(\lambda-\hat{p})}$$

and

$$D_l(k^2,l^2) = \frac{V_l(k^2l^2)}{-k^2-i\varepsilon}$$

Fort the proof of the relation (66) consider the identity

$$\frac{\partial\tilde{S}(\hat{p},\lambda)}{\partial p_\mu} = \tilde{S}(\hat{p},\lambda)\gamma^\mu\tilde{S}(\hat{p},\lambda) \quad (69)$$

Further, it is easy to verify the identity (66) by differentiating (67) over p_μ and making use of the equality (69) as well as choosing other momentum variables in (68) and assuming $q=0, p^{'}=p+q=p$. The relations of the type

$$q_\mu \tilde{\Gamma}_\mu^{l,s}(p,q)\big|_{p'^2=p^2=\lambda^2} = 0$$

follows from the definition

$$q_\mu E_{xpec.}\{\tilde{\Omega}(p_1)\gamma^\mu \tilde{\Omega}(p_2)\} = q_\mu \int_{-m}^{m}\int_{-m}^{m} d\lambda_1 d\lambda_2 \rho(\lambda_1)\rho(\lambda_2) \times$$
$$\times \tilde{S}(\hat{p}_1,\lambda_1)\gamma^\mu \tilde{S}(\hat{p}_2,\lambda_2)\frac{\delta(\lambda_1-\lambda_2)}{\rho(\lambda_1)} = \tilde{\Omega}(p_1) - \tilde{\Omega}(p_2) =$$
$$= \int_{-m}^{m} d\lambda\rho(\lambda)[\tilde{S}(\hat{p}_1,\lambda) - \tilde{S}(\hat{p}_2,\lambda)] \tag{70}$$

if $q = p_1 - p_2$.

Now let us demonstrate that the gauge invariance of the vacuum polarization diagram in the "square - root" NQED and its matrix element is given by

$$\tilde{\prod}_{\mu\nu}^{s}(k) = e^2 E_{xpec.}\{\int d^d p Tr\{\gamma^\mu \tilde{\Omega}(p+k)\gamma^\nu \tilde{\Omega}(p)\} =$$
$$= e^2 \int_{-m}^{m} d\lambda\rho(\lambda)\int d^d p Tr\{\gamma^\mu \tilde{S}(\hat{p}+\hat{k},\lambda)\gamma^\nu \tilde{S}(\hat{p},\lambda)\} \tag{71}$$

Here we have used the d-dimensional gauge invariant regularization procedure due to [31] and the definition (64). After some calculations we obtain the same structure as in (65):

$$\prod_{\mu\nu}^{s}(k) = \frac{8i\pi^{d/2}}{\Gamma(2)}\Gamma\left(2-\frac{1}{2}d\right)(k_\mu k_\nu - k^2 g_{\mu\nu}) \times$$
$$\times \int_{-m}^{m} d\lambda\rho(\lambda)\int_0^1 dx x(1-x)[\lambda^2 - k^2 x(1-x)]^{\frac{d}{2}-2} \tag{72}$$

which is manifestly gauge invariant. Calculation of the matrix elements for $\tilde{\Sigma}_l^s(p)$ and $\tilde{\Gamma}_\mu^{l,s}(p,q)$ can be carried out by the same method as in (27) and (32) where we have to change $m \to \lambda$.

In conclusion, we notice that similar modification of the Newtonian potential (1) and (5) gives rise to a finite quantum gravitational theory with the causal Green function for the graviton:

$$G_{\mu\nu,\rho\sigma}^{c}(x) = \frac{-1}{(2\pi)^4 i}\int d^4 p e^{-ipx}\tilde{\prod}_{\mu\nu,\rho\sigma}(p) \times \frac{V_l(-p^2 l^2)}{-p^2 - i\varepsilon}$$

where the projecting tensor $\tilde{\prod}_{\mu\nu,\rho\sigma}(p)$ is given by the expression:

$$\tilde{\prod}_{\mu\nu,\rho\sigma}(p) = d_{\mu\rho}(p)d_{\nu\sigma}(p) + d_{\mu\sigma}(p)d_{\nu\rho}(p) - \frac{2}{3}d_{\mu\nu}(p)d_{\rho\sigma}(p),$$

$$d_{\mu\nu}(p) = g_{\mu\nu} - p_\mu p_\nu / p^2$$

and $V_l(-p^2 l^2)$ is defined by the same formula (7). Here l should be changed by the Planck length:

$$l \to l_{Pl} = \sqrt{\frac{\hbar G_N}{c^3}} = 1.62 \times 10^{-33}\ cm$$

where G_N is the Newtonian constant.

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