



A Non-Delay versus Delay Logistic Model of Prey-Predator Interaction

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Abstract Predation is an interaction between two populations of animals in which one (the predator) hunts, captures, and kills the other (prey) for food. The main purpose of this thesis is to study and compare the behavior of prey-predator interactions in two nonlinear population models, one without delay and the other with delay. In each case, we obtained three equilibrium states: the trivial state where both populations are extinct, the endemic state where both populations exist and another state where the prey population exists but the predator population is extinct. In the model without delay, we showed that the endemic state is asymptotically stable while the trivial state and the state where the predator population is extinct are unstable. However, in the model with delay, we proved that the endemic state and the state where the predator population is extinct are asymptotically stable while the trivial state is unstable. Furthermore, we carried out numerical experiments where we compared the dynamics of the prey and predator populations. We found that in each model the prey and predator populations oscillate with time; in particular, the two populations circle around their endemic equilibrium states.

Keywords Delay, Non-Delay, Logistic, Prey-Predictor, Interaction

1. Introduction

In ecology, predation is an interaction between two populations of animals in which one (the predator) hunts, captures, and kills the other (prey) for food. Predator-prey relationship form important links in many food chains. They are also important in regulating population sizes of both predator and prey, especially when the predator relies on a single prey species. Examples of predator-prey relationship are cats and rats, and lizards and grasshoppers [1].

Four types of predation may be distinguished. Herbivores are animals that prey on green plants or their seeds and fruits; often the plants eaten are not killed but may be damaged. Typical predation occurs when carnivores prey on herbivores or other carnivores. Insect parasitism is another form of predation, in which the insect parasite lays eggs on or near the host insect, which is subsequently killed and eaten. Finally, cannibalism is a special form of predation, in which the predator and the prey are the same species. All these processes can be described with the same kind of mathematical models [2].

Mathematical models have been used extensively to build up hypothesis about what happens when two species live together either sharing the same food, occupying the same space, or preying on or parasitizing the others. The best-known models of these phenomena are the Lotka-Volterra equations, which were derived



independently [3]. The set of equations to describe the interaction between population of predators and preys are as follows.

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = -py + qxy$$

where $x(t)$ is the prey population at time t , $y(t)$ is the predator population at time t and a, b, p, q are positive constants,

The Lotka-Volterra predation model predicts oscillations between predators and prey that are called neutrally stable because the oscillations are determined by the starting conditions. Such neutral stability is the same stability shown by a frictionless pendulum and will be very susceptible to all disturbances found in natural populations.

Kapur [4] modeled a population-growth and prey-predator models in terms of delay-differential equations (DDE). The postulate is that the birth-rate at time t depends on the population size at time $t - \tau$, $\tau > 0$ and the death-rate depends on the population size at time, t . Assuming that the interaction between prey and predators species results in instantaneous loss to the prey species, but a gain to the predator species, we get the model

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = -py + qx(t - \tau)y(t - \tau)$$

The effect of predation on population has been studied theoretically and practically because it has great economic implication for man. Predator-prey relationship helps in the control of pest population (biological control) which, when applied on a large scale, will save man from the effects of chemical control i.e biomagnifications of insecticidal residues through food chain [5].

The first equation in the Lotka-Volterra model is a linear equation of population growth with predation. This can be written as a non-linear model of population growth with predation as follows;

$$\frac{dx}{dt} = ax - cx^2 - bxy,$$

where c is another positive constant.

Predation is the description of a biological interaction where a predator (an organism that is hunting) feeds on its prey (the organism that is attacked), [6]. Predators may or may not kill their prey prior to feeding on them, but the act of predation always results in the death of its prey and the eventual absorption of the prey's tissue through consumption. Other categories of consumption are herbivores (eating parts of plants) and detritivory, the consumption of dead organic material (detritus). All these consumption categories fall under the rubric of consumer-resource systems [7]. It can often be difficult to separate various types of feeding behaviors [6]. For example, some parasitic species prey on a host organism and then lay their eggs on it for their offspring to feed on it while it continues to live or on its decaying corpse after it has died. The key characteristic of predation however is the predator's direct impact on the prey population. On the other hand, detritivores simply eat dead organic material arising from the decay of dead individuals and have no direct impact on the "donor" organisms. Mathematical models have been used extensively to build up hypothesis about what happens when two species live together either sharing the same food, occupying the same space, or preying on or parasitizing the others. The best-known models of these phenomena are the Lotka- Volterra equations, which were derived independently, (Lotka [3]) in the United States and (Volterra 1926) in Italy. The set of equations to describe the interaction between populations of predator and prey are as follows,

$$\frac{dx}{dt} = ax - bxy,$$

$$\frac{dy}{dt} = -py + qxy,$$

where $x(t)$ is the prey population at time t , $y(t)$ is the predator population at time t and a, b, p, q are positive constants. The first equation in the Lotka-Volterra model is a linear equation of population growth with predation. This can be written as a nonlinear model of population growth with predation as follows;

$$\frac{dx}{dt} = ax - cx^2 - bxy,$$



where c is another positive constant.

According to [8], in ecosystems, species are connected through intricate trophic relationships defining complex networks, the so-called food chains. Understanding the structure and mechanisms underlying the formation of these complex chains is of great importance to ecology. In particular, the food chain structure provides insight into the behavior of an ecosystem under perturbations such as the extinction of existing species, [9].

2.1. Preliminaries

A wide variety of natural phenomena can be modeled by a system of two first-order autonomous systems of ordinary differential equations (ODE) of the form

$$\frac{dx}{dt} = f(x, y) \quad (2.1.1)$$

$$\frac{dy}{dt} = g(x, y) \quad (2.1.2)$$

Where f and g are differential functions in some region D of the xy -plane, called the phase plane of the system (2.1.1) and (2.1.2). Then, given t_0 and any initial point (x_0, y_0) of D , there is a unique solution $x = x(t)$, $y = y(t)$ of (2.1.1) and (2.1.2) that is defined on some open interval (a, b) containing t_0 and satisfying the initial conditions;

$$x(t_0) = x_0, \quad y(t_0) = y_0 \quad (2.1.3)$$

An equilibrium state of the system in (2.1.1) and (2.1.2) is a state (x_*, y_*) such that

$$f(x_*, y_*) = g(x_*, y_*) = 0 \quad (2.1.4)$$

For any equilibrium state (x_*, y_*) of the system, the constant-valued solutions

$$x(t) \equiv x_*, \quad y(t) \equiv y_* \quad (2.1.5)$$

satisfy equations (2.1.1) and (2.1.2). Such a solution is called an equilibrium solution of the system. The trajectory of the equilibrium solution consists of the single point (x_*, y_*) .

The linear form of the system (2.1.1) and (2.1.2) can be represented as

$$X' = AX \quad (2.1.6)$$

where A is a 2×2 matrix with constant coefficients and X is a 2 by 1 column vector.

Definition 2.1.1 (Stability of an equilibrium state (x_*, y_*))

Let $X_* = (x_*, y_*)$ be an equilibrium state of an autonomous system, and

$X = (x(t), y(t))$ denote the solution that satisfies the initial condition $X_0 = (x_0, y_0)$, where $X_0 \neq X_*$,

- We say that X_* is a stable equilibrium state when, given $\epsilon > 0$ there exists $\delta > 0$ such that; $\|X_0 - X_*\| < \delta \Rightarrow \|X(t) - X_*\| < \epsilon$ for all $t > 0$.
- If, in addition, $\lim_{t \rightarrow \infty} X(t) = X_*$ whenever $\|X_0 - X_*\| < \delta$, we call X_* an asymptotically stable equilibrium state.
- It is unstable, if it is not stable.

Theorem 2.1.1

Suppose that the matrix A in (2.1.6) has eigenvalues λ_1, λ_2 . Then the stability of any solution of (2.1.6) is determined according to the following criteria:

- If the real part of an eigenvalue $Re(\lambda_i) < 0$, $i = 1, 2$ then there is uniform and asymptotic stability.
- If $Re(\lambda_i) \leq 0$ for all $i = 1, 2$ and algebraic multiplicity equals the geometric multiplicity whenever $\lambda_i = 0$ for any i then there is uniform stability.
- If $Re(\lambda_i) > 0$ for at least one i or the algebraic multiplicity is greater than the geometric multiplicity should be, $Re(\lambda_i) = 0$, then there is instability.

Note that the algebraic multiplicity of an eigenvalue of a matrix is the multiplicity of such an eigenvalue as a root of the characteristic equation while the geometric multiplicity is the multiplicity of an eigenvector corresponding to the given eigenvalue.

The first step in investigating the stability of the system (2.1.1) and (2.1.2) at an equilibrium state (x_*, y_*) is by finding its Jacobian matrix at (x_*, y_*) . The process is called linearization [10].

We assume that the functions f and g are continuously differentiable in a neighborhood of (x_*, y_*) .



Let, $u = x - x_*$, $v = y - y_*$, so that $\frac{dx}{dt} = \frac{du}{dt}$ and $\frac{dy}{dt} = \frac{dv}{dt}$, that is, $x = u + x_*$, $y = v + y_*$, and equation (2.1.1) and (2.1.2) becomes

$$\frac{du}{dt} = f(u + x_*, v + y_*) \quad (2.1.7)$$

$$\frac{dv}{dt} = g(u + x_*, v + y_*) \quad (2.1.8)$$

Then the Taylor formula for f and g about the equilibrium state (x_*, y_*) gives

$$f(u + x_*, v + y_*) = f_x(x_*, y_*)u + f_y(x_*, y_*)v + h_1(u, v) \quad (2.1.9)$$

$$g(u + x_*, v + y_*) = g_x(x_*, y_*)u + g_y(x_*, y_*)v + h_2(u, v) \quad (2.1.10)$$

where $h_1(u, v)$ and $h_2(u, v)$ have the property that

$$\lim_{\|(u,v)\| \rightarrow 0} \frac{\|h_1(u,v)\|}{\|(u,v)\|} = \lim_{\|(u,v)\| \rightarrow 0} \frac{\|h_2(u,v)\|}{\|(u,v)\|} = 0$$

That is, in the neighborhood of (x_*, y_*) , the expression $h_1(u, v)$ and $h_2(u, v)$ are small in comparison with $\|(u, v)\|$, which is itself small. In matrix form, (2.1.9) and (2.1.10) become

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} f_x(x_*, y_*) & f_y(x_*, y_*) \\ g_x(x_*, y_*) & g_y(x_*, y_*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix} \quad (2.1.11)$$

The matrix

$$J = \begin{pmatrix} f_x(x_*, y_*) & f_y(x_*, y_*) \\ g_x(x_*, y_*) & g_y(x_*, y_*) \end{pmatrix} \quad (2.1.12)$$

Is the Jacobian Matrix of the system (2.1.1) and (2.1.2), evaluated at the equilibrium state (x_*, y_*) . The system in (2.1.11) is called the linearization of (2.1.1) and (2.1.2) about the equilibrium state (x_*, y_*) .

Theorem 2.1.2 (stability criteria for plane autonomous system)

Let X_* be an equilibrium state of the plane autonomous system (2.1.1) and (2.1.2) where $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives in a neighborhood of X_* .

- If the eigenvalues of (2.1.12) have negative real parts, then X_* is an asymptotically stable equilibrium state.
- If (2.1.12) has an eigenvalue with positive real part then X_* is an unstable equilibrium state.

2.2. The Mathematical Models

We consider two population models for a prey-predator interaction given, [3]. The first model is the model without delay which is given by

$$\frac{dx}{dt} = ax - cx^2 - bxy \quad (2.2.1)$$

$$\frac{dy}{dt} = -py + qxy \quad (2.2.2)$$

where a, b, c, p and q are all positive constants, (Rodney 1978). We shall denote the equations (2.2.1) and (2.2.2) as model A.

The second model, which has a delay [4], and is given as

$$\frac{dx}{dt} = ax - cx^2 - bxy \quad (2.2.3)$$

$$\frac{dy}{dt} = -py + qx(t - \tau)y(t - \tau) \quad (2.2.4)$$

where $\tau > 0$ is the delay. We shall label the equations (2.2.3) and (2.2.4) as model B.

The variables and parameters of models A and B are as defined below.

$y(t)$ = The number of predator at time t

$x(t)$ = The number of prey at time t

a = The growth rate of prey

b = The rate of decrease of the prey due to inhibition by the predator.

c = The rate of decrease of the prey due to insufficient food.

p = The death rate of the predators.

q = The rate of increase of the predator due to successful attacks on the prey.



2.3. Equilibrium States

2.3.1. The Equilibrium States of Model A

In this section, the existence of the equilibrium states of model A is discussed. At the equilibrium state, $\frac{dx}{dt} = \frac{dy}{dt} = 0$. Therefore, equating the left hand sides of model A to zero gives

$$ax - cx^2 - bxy = 0 \quad (2.3.1.1)$$

$$-py + qxy = 0 \quad (2.3.1.2)$$

Factorizing the equations (2.3.1.1) and (2.3.1.2), and solving

We obtained the following equilibriums for model A: $(0, 0)$, $(\frac{a}{c}, 0)$, $(\frac{p}{q}, \frac{aq-cp}{bq})$.

Thus we have proved the following lemma.

Lemma 2.3.1

Given that $a, b, c, p, q > 0$

- Then there exists an equilibrium state $(x_*, y_*) = (\frac{a}{c}, 0)$ of model A.
- If $aq > cp$, then there exists another equilibrium state $(x_*, y_*) = (\frac{p}{q}, \frac{aq-cp}{bq})$ of model A.
- There exists a trivial equilibrium state, $(x_*, y_*) = (0, 0)$ of model A.

2.3.2. The Equilibrium State of Model B

We can find the equilibrium states of the system (2.2.3) and (2.2.4), by putting $\frac{dx}{dt} = \frac{dy}{dt} = 0$ and replacing; $x(t) = \bar{x}$, $x(t - \tau) = \bar{x}$, and $y(t) = \bar{y}$, $y(t - \tau) = \bar{y}$. The equations (2.2.3) and (2.2.4) becomes

$$a\bar{x} - c\bar{x}^2 - b\bar{x}\bar{y} = 0 \quad (2.3.2.1)$$

and

$$-p\bar{y} + q\bar{x}\bar{y} = 0 \quad (2.3.2.2)$$

These equations (2.3.2.1) and (2.3.2.2) are similar to equations (2.3.1.1) and (2.3.1.2). Therefore, solving the algebraic equations (2.3.2.1) and (2.3.2.2) simultaneously for \bar{x} and \bar{y} , we obtain the following equilibrium states $(0, 0)$, $(\frac{a}{c}, 0)$ and $(\frac{p}{q}, \frac{aq-cp}{bq})$. These equilibrium states exist if and only if lemma 2.3.1 holds

2.4.1. The Stability of Equilibrium States of Model A

We now analyze the stability of each of equilibrium states of (2.2.1) and (2.2.2) found in section (2.3.1). First we shall compute the Jacobian matrix of the system (2.2.1) and (2.2.2).

Let

$$f(x, y) = ax - cx^2 - bxy$$

and

$$g(x, y) = -py + qxy$$

Then

$$f_x(x, y) = a - 2cx - by$$

$$f_y(x, y) = -bx$$

$$g_x(x, y) = qy$$

$$g_y(x, y) = -p + qx$$

Therefore, the Jacobian matrix of the system (2.2.1) and (2.2.2) is given by

$$J = \begin{pmatrix} a - 2cx - by & -bx \\ qy & -p + qx \end{pmatrix} \quad (2.4.1.1)$$

Computing the Jacobian matrix (2.4.1.1) at the equilibrium state $(0, 0)$ gives

$$J_{(0,0)} = \begin{pmatrix} a & 0 \\ 0 & -p \end{pmatrix} \quad (2.4.1.2)$$

The characteristic equation for the Jacobian matrix (2.4.1.2) is

$$\det(J_{(0,0)} - \lambda I) = 0$$

Thus



$$\begin{vmatrix} a - \lambda & 0 \\ 0 & -p - \lambda \end{vmatrix} = 0$$

or

$$(a - \lambda)(-p - \lambda) = 0,$$

either

$$a - \lambda = 0 \text{ or } -p - \lambda = 0$$

That is,

$$\lambda_1 = a > 0 \text{ or } \lambda_2 = -p < 0$$

Therefore, the equilibrium state $(0, 0)$ is unstable by theorem (2.1.2). We shall now analyze the equilibrium state $(\frac{a}{c}, 0)$ for stability. Evaluating the Jacobian matrix (2.4.1.1) at the equilibrium state $(\frac{a}{c}, 0)$ gives

$$J_{(\frac{a}{c}, 0)} = \begin{pmatrix} -a & \frac{-ab}{c} \\ 0 & -p + \frac{aq}{c} \end{pmatrix} \quad (2.4.1.3)$$

The eigenvalues for this Jacobian matrix (2.4.1.3) is obtained as follows;

$$\begin{vmatrix} -a - \lambda & \frac{-ab}{c} \\ 0 & \frac{-cp + aq}{c} - \lambda \end{vmatrix} = 0$$

or

$$(-a - \lambda) \left(\frac{-cp + aq}{c} - \lambda \right) = 0$$

This gives

$$\lambda_1 = -a < 0 \text{ or } \lambda_2 = \frac{-cp + aq}{c} = \frac{aq - cp}{c} > 0,$$

Because $aq > cp$ by lemma 2.3.1 b.

Therefore, the equilibrium state $(\frac{a}{c}, 0)$ is unstable by theorem 2.1.2

Next we shall examine the stability of the equilibrium state $(\frac{p}{q}, \frac{aq - cp}{bq})$ by evaluating the Jacobian matrix (2.4.1.1) at the equilibrium state $(\frac{p}{q}, \frac{aq - cp}{bq})$ to obtain the matrix

$$J = \begin{pmatrix} \frac{-cp}{q} & \frac{-bp}{q} \\ \frac{-aq - cp}{b} & 0 \end{pmatrix} \quad (2.4.1.4)$$

The eigenvalues of the matrix (2.4.1.4) can be obtained as follows

$$\begin{vmatrix} \frac{-cp}{q} - \lambda & \frac{-bp}{q} \\ \frac{aq - cp}{b} & -\lambda \end{vmatrix} = 0$$

or

$$-\lambda \left(\frac{-cp}{q} - \lambda \right) + \frac{bp}{q} \left(\frac{aq - cp}{b} \right) = 0$$

or

$$\lambda^2 + \frac{cp\lambda}{q} + \frac{p}{q}(aq - cp) = 0$$

Multiplying each term by q we get

$$q\lambda^2 + cp\lambda + p(aq - cp) = 0$$

And solving quadratically, we obtained the following results;

Therefore,

$$\lambda_1 < 0, \lambda_2 = \frac{-cp}{q} < 0$$

Therefore, the equilibrium state $(\frac{p}{q}, \frac{aq - cp}{bq})$ is asymptotically stable.



We shall now summarize the results obtained for the three equilibrium states of model A in the following lemma

Lemma 2.4.1

Given $a, b, c, p, q > 0$ in (2.2.1) and (2.2.2),

- The equilibrium state $(0, 0)$ is unstable.
- The state $(\frac{a}{c}, 0)$ is also unstable equilibrium state.
- The equilibrium state $(\frac{p}{q}, \frac{aq-cp}{bq})$ is asymptotically stable, provided $aq > cp$.

2.4.2. The Stability of Equilibrium State of Model B

The stability of the equilibrium states of the prey-predator model of the system (2.2.3) and (2.2.4) can be analyzed as follow:

We start with the equilibrium state $(\bar{x}, \bar{y}) = (0, 0)$

Let

$$\begin{aligned} u &= x - \bar{x} \text{ and } v = y - \bar{y} \\ &= x - 0 \quad = y - 0 \\ &= x \quad = y \end{aligned}$$

Therefore,

$$\frac{du}{dt} = \frac{dx}{dt} \text{ and } \frac{dv}{dt} = \frac{dy}{dt}$$

Substituting into the system (2.2.3) and (2.2.4) we get

$$\frac{du}{dt} = au - au^2 - buv \tag{2.4.2.1}$$

$$\frac{dv}{dt} = -pv + qu(t - \tau)v(t - \tau) \tag{2.4.2.2}$$

Neglecting squares and products of $u(t), v(t), u(t - \tau)$ and $v(t - \tau)$ from the system (2.4.2.1) and (2.4.2.2), we obtain the Jacobian matrix

$$J = \begin{pmatrix} a & 0 \\ 0 & -p \end{pmatrix}$$

This Jacobian matrix is the same with the one obtained from model A. Therefore, the equilibrium state $(0, 0)$ is unstable, see section 2.4.1

We consider the next equilibrium state $(\bar{x}, \bar{y}) = (\frac{a}{c}, 0)$

Let

$$u = x - \frac{a}{c} \text{ and } v = y - 0$$

That is,

$$x = u + \frac{a}{c}, \quad y = v$$

Therefore,

$$\frac{dx}{dt} = \frac{du}{dt} \text{ and } \frac{dy}{dt} = \frac{dv}{dt}$$

Again, substituting into the system (2.2.3) and (2.2.4), equation (2.2.3) gives

$$\frac{du}{dt} = a\left(u + \frac{a}{c}\right) - c\left(u + \frac{a}{c}\right)^2 - b\left(u + \frac{a}{c}\right)v \tag{2.4.2.3}$$

Expanding the brackets and simplifying the expression (2.4.2.3), we have

$$\begin{aligned} \frac{du}{dt} &= au + \frac{a^2}{c} - c\left(u^2 + \frac{2au}{c} + \left(\frac{a}{c}\right)^2\right) - \frac{bav}{c} - buv \\ &= (a - 2a)u - \frac{abv}{c} - cu^2 - buv \end{aligned}$$

or

$$\frac{du}{dt} = -au - \frac{abv}{c} - cu^2 - buv \tag{2.4.2.4}$$

Equation (2.2.4) gives

$$\frac{dv}{dt} = -pv + q\left(u(t - \tau) + \frac{a}{c}\right)v(t - \tau)$$

or



$$\frac{dv}{dt} = -pv + \frac{aq}{c} v(t - \tau) + qu(t - \tau)v(t - \tau) \quad (2.4.2.5)$$

Neglecting the squares and products of $u(t)$, $v(t)$, $u(t - \tau)$ and $v(t - \tau)$ in equation (2.4.2.4) and (2.4.2.5) we get the linear delay-differential equations

$$\frac{du}{dt} = -au - \frac{abv}{c} \quad (2.4.2.6)$$

$$\frac{dv}{dt} = -pv + \frac{aq}{c} v(t - \tau) \quad (2.4.2.7)$$

To find the characteristic equation and eigenvalues,

Let

$$u(t) = Ae^{\lambda t}, \quad v(t) = Be^{\lambda t}$$

This implies that,

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} A \\ B \end{pmatrix} \quad (2.4.2.8)$$

where A and B are non-zero real numbers while λ may be real or complex.

Then,

$$\frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \lambda e^{\lambda t} \begin{pmatrix} A \\ B \end{pmatrix} \quad (2.4.2.9)$$

Substituting equations (2.4.2.8) and (2.4.2.9) into equation (2.4.2.6) and (2.4.2.7), we get

$$A\lambda e^{\lambda t} = aAe^{\lambda t} - \frac{ab}{c} Be^{\lambda t} \quad (2.4.2.10)$$

and

$$B\lambda e^{\lambda t} = -pBe^{\lambda t} + \frac{aq}{c} Be^{\lambda(t-\tau)} \quad (2.4.2.11)$$

Multiplying equation (2.4.2.10) by $e^{-\lambda t}$, we get

$$A\lambda = -aA - \frac{ab}{c} B \quad (2.4.2.12)$$

We shall divide (2.4.2.11) by $Be^{\lambda t}$ to get

$$\lambda = -p + \frac{aq}{c} e^{-\lambda\tau} \quad (2.4.2.13)$$

Multiplying equation (2.4.2.13) by A we get

$$A\lambda = -pA + \frac{aq}{c} Ae^{-\lambda\tau} \quad (2.4.2.14)$$

Subtracting equation (2.4.2.14) from (2.4.2.12) gives

$$(p - a)A - \frac{ab}{c} B - \frac{aq}{c} Ae^{-\lambda\tau} = 0$$

This gives

$$\frac{abB + aqAe^{-\lambda\tau}}{c} = (p - a)A$$

That is,

$$aqAe^{-\lambda\tau} = cA(p - a) - abB$$

or

$$e^{-\lambda\tau} = \frac{cA(p - a) - abB}{aqA}$$

Taking the natural logarithm of both sides we have

$$-\lambda\tau = \ln\left(\frac{cA(p - a) - abB}{aqA}\right)$$

This gives

$$\lambda = \frac{-1}{\tau} \ln\left(\frac{cA(p - a) - abB}{aqA}\right) \quad (2.4.2.15)$$

where $A \neq 0$ and $B \neq 0$. All other parameters are positive. This solution (2.4.2.15) is undefined if $\frac{cA(p - a) - abB}{aqA} < 0$, this solution (2.4.2.15) exists if and only if $cA(p - a) - abB$ and aqA have the same arithmetic signs. We shall discuss the solution of (2.4.2.15) at the various values of A and B for which the solution is defined.

Case I ($A < 0$) and ($B < 0$)

Let $\alpha = cA(p - a) - abB$ and $\beta = aqA$



and so

$$\beta < 0$$

and

$$\alpha < 0 \text{ if } p > a \text{ and } cA(p - a) > abB$$

Thus

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{\alpha}{\beta} \right) < 0$$

Case II ($A < 0$ and $B > 0$)

Again

$$\beta < 0$$

and

$$\alpha < 0 \text{ if } p > a \text{ and } abB > cA(p - a)$$

This gives

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{\alpha}{\beta} \right) < 0$$

Case III ($A > 0$ and $B < 0$)

$$\beta > 0$$

and

$$\alpha > 0 \text{ if } p < a$$

or

$$\alpha > 0 \text{ if } p > a \text{ and } abB > cA(p - a)$$

Thus

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{\alpha}{\beta} \right) < 0$$

Case IV ($A > 0$ and $B > 0$)

we get

$$\beta > 0$$

and

$$\alpha > 0 \text{ if } p > a \text{ and } abB < cA(p - a)$$

Again

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{\alpha}{\beta} \right) < 0$$

Therefore, the equilibrium state $\left(\frac{a}{c}, 0\right)$ is asymptotically stable if the four cases discussed above hold. We shall consider the situation in which equation (2.4.2.10) and (2.4.2.11) have complex solutions.

To this end, let

$$\lambda = r + is \tag{2.4.2.16}$$

Substituting (2.4.2.16) into (2.4.2.12) and (2.4.2.14) we have

$$A(r + is) = -aA - \frac{abB}{c} \tag{2.4.2.17}$$

and

$$A(r + is) = -pA + \frac{aq}{c} Ae^{-\tau(r+is)}$$

or

$$A(r + is) = -pA + \frac{aq}{c} Ae^{-\tau r} (\cos(\tau s) - i \sin(\tau s)) \tag{2.4.2.18}$$

Subtracting (2.4.2.18) from (2.4.2.17) gives

$$(p - a)A - \frac{ab}{c} B - \frac{aq}{c} Ae^{-\tau r} (\cos(\tau s) - i \sin(\tau s)) = 0$$

or

$$\frac{aq}{c} Ae^{-\tau r} \cos(\tau s) - \frac{iaq}{c} Ae^{-\tau r} \sin(\tau s) = \frac{cA(p - a) - abB}{c}$$



That is,

$$aqAe^{-\tau r} \cos(\tau s) - iaqAe^{-\tau r} \sin(\tau s) = cA(p - a) - abB \quad (2.4.2.19)$$

Equating the real and imaginary parts of equation (2.4.2.19) we get

$$aqAe^{-\tau r} \cos(\tau s) = cA(p - a) - abB \quad (2.4.2.20)$$

and

$$aqAe^{-\tau r} \sin(\tau s) = 0 \quad (2.4.2.21)$$

In equation (2.4.2.21),

$$\sin(\tau s) = 0, \text{ since } a, q > 0, A \neq 0 \text{ and } e^{-\tau r} \neq 0$$

This implies that

$$\tau s = n\pi, \quad n = 0, 1, 2 \dots$$

From equation (2.4.2.20), if $n = 0$

$$\cos(\tau s) = \cos 0 = 1$$

Putting $\cos(\tau s) = 1$ in equation (2.4.2.20) gives

$$aqAe^{-\tau r} = cA(p - a) - abB$$

$$e^{-\tau r} = \frac{cA(p - a) - abB}{aqA}$$

Taking the natural logarithm of both sides we get

$$-\tau r = \ln\left(\frac{cA(p - a) - abB}{aqA}\right)$$

That is,

$$r = \frac{-1}{\tau} \ln\left(\frac{cA(p - a) - abB}{aqA}\right) \quad (2.4.2.22)$$

This equation (2.4.2.22) is the same as (2.4.2.15) which we discussed before.

If $n = 1$

$$\cos(\tau s) = \cos \pi = -1$$

If we put $\cos(\tau s) = -1$ in equation (2.4.2.20), it will violate the condition imposed on the solution (2.4.2.15).

Therefore, we shall only consider the values of n for which

$$\cos(\tau s) = 1 \text{ (i.e. } n = 0, 2, 4). \text{ Hence, the equilibrium state } \left(\frac{a}{c}, 0\right) \text{ is asymptotically stable as earlier discussed.}$$

Lastly, we shall investigate the stability of the equilibrium state $\left(\frac{p}{q}, \frac{aq - cp}{bq}\right)$ of the system (2.2.3) and (2.2.4).

Let

$$u = x - \frac{p}{q} \text{ and } v = y - \frac{a}{b} + \frac{cp}{bq}$$

That is,

$$x = u + \frac{p}{q}, \quad y = v + \frac{a}{b} - \frac{cp}{bq} \quad (2.4.2.23)$$

Then,

$$\frac{dx}{dt} = \frac{du}{dt}, \quad \frac{dy}{dt} = \frac{dv}{dt} \quad (2.4.2.24)$$

Substituting (2.4.2.23) and (2.4.2.24) into (2.2.3) and (2.2.4) we get the following, equation (2.2.3) gives

$$\frac{du}{dt} = a\left(u + \frac{p}{q}\right) - c\left(u + \frac{p}{q}\right)^2 - b\left(u + \frac{p}{q}\right)\left(v + \frac{a}{b} - \frac{cp}{bq}\right)$$

or

$$\begin{aligned} \frac{du}{dt} &= au + \frac{ap}{q} - c\left(u^2 + \frac{2pu}{q} + \left(\frac{p}{q}\right)^2\right) - b\left(uv + \frac{au}{b} - \frac{cpu}{bq} + \frac{pv}{q} + \frac{ap}{bq} - \frac{cp^2}{bq^2}\right) \\ &= au + \frac{ap}{q} - cu^2 - \frac{2cpu}{q} - \frac{cp^2}{q^2} - buv - au + \frac{cpu}{q} - \frac{bpv}{q} - \frac{ap}{q} + \frac{cp^2}{q^2} \end{aligned}$$

Therefore,

$$\frac{du}{dt} = -\frac{cpu}{q} - \frac{bpv}{q} - cu^2 - buv \quad (2.4.2.25)$$

Also, equation (2.2.4) gives



$$\frac{dv}{dt} = -p\left(v + \frac{a}{b} - \frac{cp}{bq}\right) + q\left(u(t-\tau) + \frac{p}{q}\right)\left(v(t-\tau) + \frac{a}{b} - \frac{cp}{bq}\right)$$

or

$$\begin{aligned} \frac{dv}{dt} &= -pv - \frac{ap}{b} + \frac{cp^2}{bq} + \\ & q\left(u(t-\tau)v(t-\tau) + \frac{a}{b}(u(t-\tau)) - \frac{cpu(t-\tau)}{bq} - \frac{pv(t-\tau)}{q} + \frac{ap}{bq} - \frac{cp^2}{bp^2}\right) \\ &= -pv - \frac{ap}{b} + \frac{cp^2}{bq} + qu(t-\tau)v(t-\tau) + \frac{aq}{b}u(t-\tau) - \frac{cp}{b}u(t-\tau) \\ &+ pv(t-\tau) + \frac{ap}{b} - \frac{cp^2}{bq} \end{aligned}$$

Therefore,

$$\frac{dv}{dt} = -pv + \frac{aq-cp}{b}u(t-\tau) + pv(t-\tau) + qu(t-\tau)v(t-\tau) \quad (2.4.2.26)$$

Neglecting the squares and products of $u(t)$, $v(t)$, $u(t-\tau)$ and $v(t-\tau)$ in equation (2.4.2.25) and (2.4.2.26) we get the linear delay-differential equations

$$\frac{du}{dt} = -\frac{cp}{q}u(t) - \frac{bp}{q}v(t) \quad (2.4.2.27)$$

and

$$\frac{dv}{dt} = -pv(t) + \frac{aq-cp}{b}u(t-\tau) + pv(t-\tau) \quad (2.4.2.28)$$

Substituting (2.4.2.8) and (2.4.2.9) into (2.4.2.27) and (2.4.2.28), we obtain

$$A\lambda e^{\lambda t} = -\frac{cpAe^{\lambda t}}{q} - \frac{bpBe^{\lambda t}}{q} \quad (2.4.2.29)$$

and

$$B\lambda e^{\lambda t} = -pBe^{\lambda t} + \frac{aq-cp}{b}Ae^{\lambda(t-\tau)} + pBe^{\lambda(t-\tau)} \quad (2.4.2.30)$$

Dividing equation (2.4.2.29) by $e^{\lambda t}$ we get

$$A\lambda = -\frac{cp}{q}A - \frac{bp}{q}B \quad (2.4.2.31)$$

We shall also divide equation (2.4.2.30) by $e^{\lambda t}$ we get

$$B\lambda = -pB + \frac{aq-cp}{b}Ae^{-\lambda\tau} + pBe^{-\lambda\tau} \quad (2.4.2.32)$$

Multiplying equation (2.4.2.31) by B

$$AB\lambda = -\frac{cp}{q}AB - \frac{bp}{q}B^2 \quad (2.4.2.33)$$

Multiplying equation (2.4.2.32) by A

$$AB\lambda = -pAB + \frac{aq-cp}{b}A^2e^{-\lambda\tau} + pABe^{-\lambda\tau}$$

or

$$AB\lambda = -pAB + \left(\frac{aq-cp}{b}A^2 + pAB\right)e^{-\lambda\tau} \quad (2.4.2.34)$$

Subtracting equation (2.4.2.34) from equation (2.4.2.33) gives

$$pBA - \frac{cp}{q}AB - \frac{bp}{q}B^2 - \left(\frac{aq-cp}{b}A^2 + pAB\right)e^{-\lambda\tau} = 0$$

That is,

$$\left(\frac{(aq-cp)A^2 + bpAB}{b}\right)e^{-\lambda\tau} = \frac{pqAB - cpAB - bpB^2}{q}$$

This gives

$$e^{-\lambda\tau} = \frac{bp(AB(q-c) - bB^2)}{q((aq-cp)A^2 + bpAB)}$$

Taking the natural logarithm of both sides we obtain

$$-\lambda\tau = \ln\left(\frac{bp(AB(q-c) - bB^2)}{q((aq-cp)A^2 + bpAB)}\right)$$



This gives

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{bp(AB(q-c) - bB^2)}{q((aq-cp)A^2 + bpAB)} \right) \quad (2.4.2.35)$$

The solution (2.4.2.35) is undefined if $\left(\frac{bp(AB(q-c) - bB^2)}{q((aq-cp)A^2 + bpAB)} \right) < 0$

We shall discuss the solution (2.4.2.35) for various values of A and B as before.

Case I ($A < 0$) and ($B < 0$)

Let

$$\mu = bp(AB(q-c) - bB^2)$$

and

$$\gamma = q((aq-cp)A^2 + bpAB)$$

This gives

$$\gamma > 0, \text{ since } A < 0 \text{ and } B < 0$$

and

$$\mu > 0, \text{ if } q > c \text{ and } AB(q-c) > bB^2$$

Thus

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{\mu}{\gamma} \right) < 0$$

Case II ($A < 0$ and $B > 0$)

Then

$$\gamma > 0, \text{ if } (aq-cp)A^2 > bpAB$$

and

$$\mu > 0, \text{ if } q < c \text{ and } AB(q-c) > bB^2$$

also

$$\gamma < 0, \text{ if } (aq-cp)A^2 < bpAB$$

and

$$\mu < 0, \text{ if } q > c$$

With these conditions,

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{\mu}{\gamma} \right) < 0$$

Case III ($A > 0$ and $B < 0$)

We get similar conditions as in case II.

Case IV ($A > 0$ and $B > 0$)

Then

$$\gamma > 0$$

and

$$\mu > 0, \text{ if } q > c \text{ and } AB(q-c) > bB^2$$

Thus

$$\lambda = \frac{-1}{\tau} \ln \left(\frac{\mu}{\gamma} \right) < 0$$

Therefore, with the four cases discussed above, the equilibrium state $\left(\frac{p}{q}, \frac{aq-cp}{bq} \right)$ is asymptotically stable.

Again, we shall consider the situation where the equilibrium state $\left(\frac{p}{q}, \frac{aq-cp}{bq} \right)$ gives complex eigenvalues.

Substituting (2.4.16) into (2.4.2.33) and (2.4.2.34) gives

$$AB(r + is) = \frac{-cp}{q} AB - \frac{bp}{q} B^2 \quad (2.4.2.36)$$

and

$$AB(r + is) = -pAB + \left(\frac{aq-cp}{b} A^2 + pAB \right) e^{-\tau(r+is)} \quad (2.4.2.37)$$

Substituting (2.4.2.36) into (2.4.2.37)



$$\frac{-cp}{q}AB - \frac{bp}{q}B^2 = -pAB + \left(\frac{(aq - cp)A^2 + bpAB}{b}\right)e^{-\tau r}(\cos(\tau s) - i \sin(\tau s))$$

or

$$pAB - \frac{cpAB}{q} - \frac{bpB^2}{q} = \left(\frac{(aq - cp)A^2 + bpAB}{b}\right)e^{-\tau r} \cos(\tau s) - i \left(\frac{(aq - cp)A^2 + bpAB}{b}\right)e^{-\tau r} \sin(\tau s)$$

Equating the real and imaginary parts of both sides we obtain

$$\left(\frac{(aq - cp)A^2 + bpAB}{b}\right)e^{-\tau r} \cos(\tau s) = pAB - \frac{cpAB}{q} - \frac{bpB^2}{q} \quad (2.4.2.38)$$

and

$$\left(\frac{(aq - cp)A^2 + bpAB}{b}\right)e^{-\tau r} \sin(\tau s) = 0 \quad (2.4.2.39)$$

In equation (2.4.2.39),

$$\sin(\tau s) = 0, \text{ since } e^{-\tau r} \neq 0 \text{ and } \left(\frac{(aq - cp)A^2 + bpAB}{b}\right) \neq 0$$

This implies that

$$\tau s = n\pi, \quad n = 0, 1, 2, \dots$$

Take $n = 0$, then $\tau s = 0$

This implies that

$$\cos(\tau s) = \cos 0 = 1$$

Putting $\cos(\tau s) = 1$ in equation (2.4.2.38) gives

$$\left(\frac{(aq - cp)A^2 + bpAB}{b}\right)e^{-\tau r} = \frac{pqAB - cpAB - bpB^2}{q}$$

or

$$e^{-\tau r} = \frac{bp(AB(q - c) - bB^2)}{q((aq - cp)A^2 + bpAB)}$$

That is,

$$r = \frac{-1}{\tau} \ln \left(\frac{bp(AB(q - c) - bB^2)}{q((aq - cp)A^2 + bpAB)} \right)$$

which is same as equation (2.4.2.35).

Thus, we have proved the following result:

Lemma 2.4.2

Given that $A \neq 0$, $B \neq 0$ and $a, b, c, p, q > 0$

a. The equilibrium state $(0, 0)$ is unstable.

b. The state $\left(\frac{a}{c}, 0\right)$ is asymptotically stable if $cA(p - a) - abB$ and aqA have the same arithmetic signs

c. The state $\left(\frac{p}{q}, \frac{aq - cp}{bq}\right)$ is asymptotically stable if $bp(AB(q - c) - bB^2)$ and $q((aq - cp)A^2 + bpAB)$ have the same arithmetic signs.

2.5. The Numerical Method.

The fourth order Runge-Kutta method (or RK_4 method) for ordinary differential equation (2.1.1) and (2.1.2) with initial conditions (2.1.3) is given as follows

$$x_{n+1} = x_n + \frac{h}{6}(M_1 + 2M_2 + 2M_3 + M_4) \quad (2.5.1)$$

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \quad (2.5.2)$$

where

$$M_1 = f(x_n, y_n)$$

$$M_2 = f\left(x_n + \frac{1}{2}hM_1, y_n + \frac{1}{2}hK_1\right)$$

$$M_3 = f\left(x_n + \frac{1}{2}hM_2, y_n + \frac{1}{2}hK_2\right)$$

$$M_4 = f(x_n + hM_3, y_n + hK_3)$$



and

$$K_1 = g(x_n, y_n)$$

$$K_2 = g\left(x_n + \frac{1}{2}hM_1, y_n + \frac{1}{2}hK_1\right)$$

$$K_3 = g\left(x_n + \frac{1}{2}hM_2, y_n + \frac{1}{2}hK_2\right)$$

$$K_4 = g(x_n + hM_3, y_n + hK_3)$$

h is the step size. We can use this method to solve the system of equation (2.2.1)- (2.2.2) and (2.2.3) - (2.2.4). From the data obtained, some graphs will be plotted to study the dynamics of the prey and predators in the two models.

In order to carry out the above task, the parameters: a, b, c, p, q and τ as defined in section (2.2) are assigned specific hypothetical values in table 1 below. The computer program written in Java to solve these equations, are presented as in Appendix B

Table 1: Tables of Parameter Values for the Numerical Experiment

Parameters	a	b	c	p	q	τ	$x(0)$	$y(0)$
Model A	4	0.5	0.02	4.25	0.6	0	20	10
Model B	4	0.5	0.02	4.25	0.6	1	20	10

As shown in table 1 above, the numerical experiments are meant to study the following cases.

1. The dynamics of prey populations in model A and model B.
2. The dynamics of predator populations in model A and model B.
3. The dynamics of prey and predator populations in model A.
4. The dynamics of prey and predator populations in model B.

The tables generated from various parameters values in Table 1 are presented in Appendix A (Tables A1 – A4) and figures obtained from the corresponding table are given in section 4.2.

3. Results

3.1 The Analytical Results

In the analysis of the two models, we have been able to obtain the following results:

Lemma 2.3.1

Given that $a, b, c, p, q > 0$

(a) Then there exists an equilibrium state $(x_*, y_*) = \left(\frac{a}{c}, 0\right)$.

(b) If $aq > cp$, then there exists another equilibrium state $(x_*, y_*) = \left(\frac{p}{q}, \frac{aq - cp}{bq}\right)$.

(c) There exists a trivial equilibrium state, $(x_*, y_*) = (0, 0)$.

In each model, we studied the stability of these equilibrium state and obtained the following results.

Lemma 2.4.1

Given $a, b, c, p, q > 0$ in model A,

(a)The equilibrium state $(0, 0)$ is unstable.

(b)The state $\left(\frac{a}{c}, 0\right)$ is also unstable equilibrium state.

(c)The equilibrium state $\left(\frac{p}{q}, \frac{aq - cp}{bq}\right)$ is asymptotically stable, provided $aq > cp$.

Lemma 2.4.2

Given that $A \neq 0, B \neq 0$ and $a, b, c, p, q > 0$ in model B,

(a)The equilibrium state $(0, 0)$ is unstable.

(b)The state $\left(\frac{a}{c}, 0\right)$ is asymptotically stable if $cA(p - a) - abB$ and aqA have the same arithmetic signs

(c)The state $\left(\frac{p}{q}, \frac{aq - cp}{bq}\right)$ is asymptotically stable if $bp(AB(q - c) - bB^2)$ and $q\left((aq - cp)A^2 + bpAB\right)$ have the same arithmetic signs.

2.2 Numerical Results

We used the data in section 2.5 to plot the following graphs in order to study the dynamics of the prey and predator in the two models.



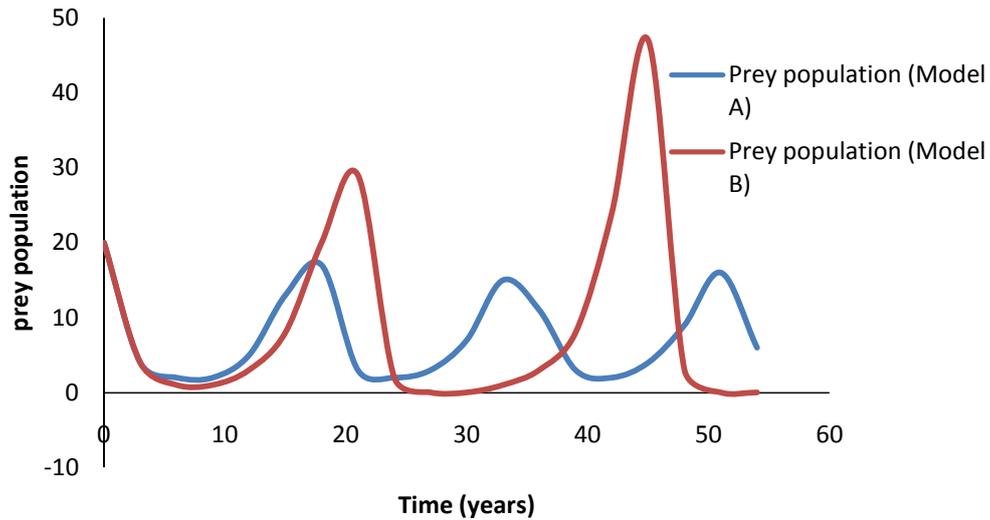


Figure 1: Graph of prey population against time in model A where $\tau = 0$ and model B where $\tau = 1$ (Numerical solution is as shown on table A1)

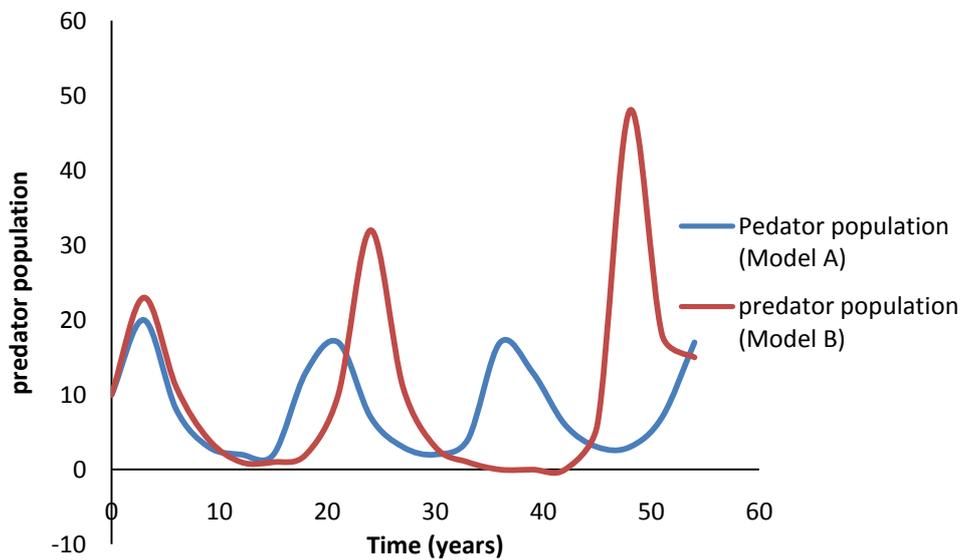


Figure 2: Graph of predator population against time, in model A where $\tau = 0$ and model B where $\tau = 1$ (Numerical solution is as shown on table A2)

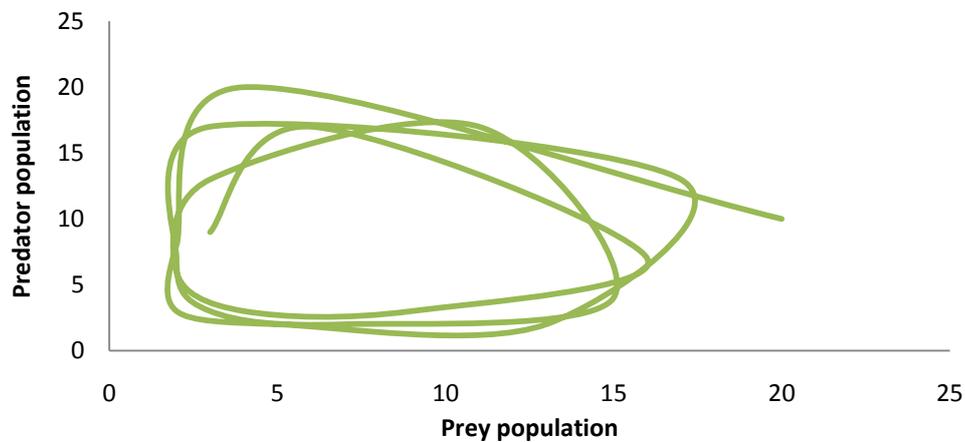


Figure 3: Graph of predator population against prey population in Model A where $\tau = 0$ (Numerical solution is as shown on table A3)

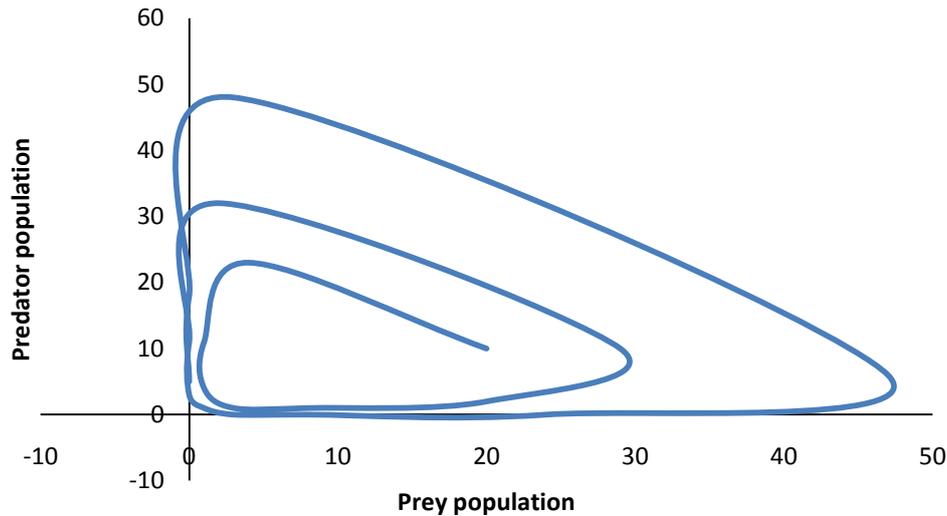


Figure 4: Graph of predator population against prey population in Model B where $\tau = 1$ (Numerical solution is as shown on table A4).

4. Discussion

4.1. Introduction

In this chapter, we discuss the analytical results and the results of our numerical experiments carried out with model A and model B presented in section 2.2. The parameter values for the experiments are shown in table 2.5. The graph plotted using these values are shown in figures 1 – 4 of section 3.2 above.

4.2. Discussion of the Analytical Results in model A and model B.

Lemma 2.3.1 shows that the prey and predator populations may attain three states, in the absence of a delay. However, the endemic state appears to be stable, meaning that the two populations will co-exist ultimately. For the prey-predator interaction with delay, it is seen from lemma 2.3.2 that the populations will either co-exist or the predator population will die out ultimately, leaving prey population. In each case the co-existence of the two populations is possible if the density dependent effects in the prey are less than the prey consumption rate by the predator.

4.3. Discussion of the Numerical Results in model A and model B.

In our numerical experiments, the following results were obtained.

4.3.1. Experiment One

Here, we studied the population of prey in model A where there is no delay *i. e* ($\tau = 0$), and the prey population in model B with delay *i. e* ($\tau = 1$). Figure 1, shows that the prey populations in both models oscillate with time. However, the prey population in model B oscillates with higher amplitude than that in model A.

4.3.2. Experiment Two

In this experiment, we studied the predator populations in model A (*i. e* $\tau = 0$) and model B where $\tau = 1$. The result in figure 2 shows that the predator populations in each model oscillate. However, the population in model A oscillates with smaller amplitudes than that in model B.

4.3.3. Experiment Three

In this experiment, we studied the dynamics of prey and predator populations in model A where $\tau = 0$. The simulation in figure 3 shows that the two populations cycle round the endemic state (7.08, 7.72).

4.3.4 Experiment Four

Here, we studied the interaction between prey and predator populations in model B where $\tau = 1$. The result in figure 4 shows that the population oscillates about the equilibrium state (7.08, 7.72).



5. Conclusion

In this research work, we have been able to study and compare two nonlinear prey-predator models, one with delay and the other without delay. Predator is recognized as one of the organizing forces in community ecology, though ecologists continue to explore its various implications. Predators are very important in controlling pests in agriculture. Natural predators are an environmentally friendly and sustainable way of reducing damage to crops, and are one alternative to the use of chemical agents such as pesticides. The two models and their parameters are shown in section 2.2. We obtained all the equilibrium states of the two models and studied the stability of these equilibrium states using linearization approach. In the model without delay, we showed that the trivial state and the state where prey population exists with predator population being extinct are unstable while the endemic state is locally asymptotically stable. However, in the model with delay, we proved that the endemic state and the state where prey population exists with predator population being extinct are locally asymptotically stable while the trivial state is unstable. This confirms the role delay play as stabilizers in prey-predator interactions. Numerical solutions of the two models using hypothetical data agree with the qualitative results.

APPENDIX A

List of Tables

Table A 1: The Numerical Solution for Model A and Model B (Parameter Values are as in Experiment 1 of Section 2.5)

Time	Prey population (Model A)	Prey population (Model B)
0	20	20
3	4	4
6	2	1
9	2	1
12	5	3
15	13	8
18	17	20
21	3	29
24	2	2
27	3	0
30	7	0
33	15	1
36	11	3
39	3	8
42	2	24
45	4	47
48	9	3
52	16	0
54	6	0

Table A2: The Numerical Solution for Model A and Model B (Parameter Values are as in Experiment 2 of Section 2.5)

Time	Predator populations (Model A)	Predator populations (Model B)
0	10	10
3	20	23
6	8	11
9	3	4
12	2	1
15	2	1



18	13	2
21	17	10
24	7	32
27	3	11
30	2	3
33	4	1
36	17	0
39	13	0
42	6	0
45	3	6
48	3	48
51	7	18
54	17	5

Table A3: The Numerical Solution for Model A (Parameter Values are as in Experiment 3 of Section 2.5)

Time	Prey population	Predator population
0	20	10
3	4	20
6	2	8
9	2	3
12	5	2
15	13	2
18	17	13
21	3	17
24	2	7
27	3	3
30	7	2
33	15	4
36	11	17
39	3	13
42	2	6
45	4	3
48	9	3
51	16	7
54	6	17

Table A4: The Numerical Solution for Model B (Parameter Values are as in Experiment 4 of Section 2.5)

Time	Prey population	Predator population
0	20	10
3	4	23
6	1	11
9	1	4
12	3	1
15	8	1
18	20	2
21	29	10
24	2	32
27	0	11
30	0	3



33	1	1
36	3	0
39	8	0
42	24	0
45	47	6
48	3	48
51	0	18
54	0	5

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