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The Fekete-Szegö Problem for the Certain Class of Analytic and Univalent Functions

Nizami Mustafa*, Muherrem C. Gündüz

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey
*Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

Abstract In this paper, we introduce and investigate new subclasses of analytic and univalent functions in the open unit disk. We give upper bound estimates for the Fekete-Szegö functional of the function belonging to these classes. In the study, special cases of these classes are also discussed.

Keywords Analytic function, univalent function, Fekete-Szegö problem, coefficient bound.
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1. Introduction and preliminaries

Let $U=\{z\in\mathbb{C}:\ |z|<1\}$ and $H(U)$ be the analytic functions in U. Denote by A the subclass of $H(U)$ functions f given by the following expansion series

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}; \tag{1.1}$$

consequently, normalized by $f(0)=0=f'(0)-1$. Also, let's S be the subclass of A, consisting also univalent functions.

It is well known that a function $f\in S$ is called starlike in U if $f(U)$ is starlike (with respect to the origin) and convex in U if $f(U)$ is convex. Starlike and convex in U function classes are denoted by S^* and C, respectively, and given below (see [5, 7, 18])

$$S^*=\left\{f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in U\right\}$$

and

$$C=\left\{f\in S:\ \mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)}\right)>0,\ z\in U\right\}.$$

The classes S^* and C are important and well investigated subclasses of S.

Some of the important and well investigated subclasses of S are the class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(\alpha\in[0,1))$, and given below

$$S^{*}(\alpha)=\left\{ f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in U\right\}$$

and

$$C(\alpha)=\left\{ f\in S:\ \mathrm{Re}\left(\frac{(zf'(z))'}{f'(z)}\right)>\alpha,\ z\in U\right\}.$$

The function classes $S^{*}(\alpha)$ and $C(\alpha)$ have been investigated rather extensively in [8, 17, 19, 20] and the references therein.

It is easy to see that $S^{*}(0)=S^{*}$ and $C(0)=C$, $S^{*}(\alpha)\subset S^{*}$ and $C(\alpha)\subset C$ for each $\alpha\in[0,1)$. Also, $f\in C\left(f\in C(\alpha)\right)\Leftrightarrow zf'\in S^{*}\left(zf'\in S^{*}(\alpha)\right)$ [5].

Among the important tools in the theory of analytic functions are Hankel determinant, which defined by coefficients of the function $f\in S$ [3]. The Hankel determinants $H_q(n)$, $n=1,2,3,...,$ $q=1,2,3,...$ of the function $f\in S$ are defined by (see [15])

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}, a_1=1.$$

Generally, this determinant was investigated by researchers with $q=2$. It is well known that the functional $H_2(1)=a_3-a_2^2$ is known as the Fekete-Szegö functional and one usually considers the further generalized functional $H_2(1,\mu)=a_3-\mu a_2^2$, where μ is a real number (see [6]). Finding estimating for the upper bound of $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö problem, in the theory of analytic functions.

In 1969 Koegh and Merkes [9] solved the Fekete-Szegö problem for the classes of starlike and convex functions for some real μ. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions ([2, 4, 11, 12, 14 -16, 21]). One can see the Fekete-Szegö problem for the classes of starlike functions of order α and convex functions of order α in special cases in the paper of Orhan et al. [16].

The object of this paper is to find the upper bound for the Fekete-Szegö functional for the subclasses of univalent functions defined as follows.

Definition 1.1. *A function* $f\in S$ *given by (1.1) is said to be in the class* $M_\beta(\alpha)$, $\beta\geq 0$, $\alpha\geq 0$ *if the following condition is satisfied*

$$\mathrm{Re}\left[(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}\right]>\alpha,\ z\in U.$$

Definition 1.2. *A function* $f \in S$ *given by (1.1) is said to be in the class* M_β, $\beta \ge 0$ *if the following condition is satisfied*

$$\operatorname{Re}\left[(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}\right]>0,\ z\in U.$$

Note 1.1. It is clear that $M_0(\alpha)=S^*(\alpha)$, $M_1(\alpha)=C(\alpha)$.

To prove our main results, we shall require the following lemmas related the functions with positive real part (see e. g. [1, 10]).

Denote by P the set of functions p analytic in U with expansion series

$$p(z)=1+\sum_{n=1}^{\infty}p_n z^n$$

and satisfying $\operatorname{Re} p(z)>0$ for $z\in U$.

Lemma 1.1. *Let* $p\in\mathrm{P}$, *then* $|p_n|\le 2$ *for every* $n=1,2,3,\ldots$. *These inequalities are sharp for each* $n=1,2,3,\ldots$.

Moreover,

$$2p_2=p_1^2+(4-p_1^2)x,$$

$$4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)(1-|x|^2)z$$

for some complex x, z *with* $|x|\le 1$, $|z|\le 1$.

2. Fekete-Szegö problem for the class $M_\beta(\alpha)$

In this section, we investigate Fekete-Szegö problem for the function classes defined by Definition 1.1.

Firstly we prove the following theorem on upper bound of the Fekete-Szegö functional $|a_3-\mu a_2^2|$ when $\mu\in\mathbb{C}$ for the function belonging to the class $M_\beta(\alpha)$.

Theorem 2.1. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\beta(\alpha)$, $\beta\ge 0$, $\alpha\in[0,1)$ *and* $\mu\in\mathbb{C}$. *Then,*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{4(1-\alpha)^2}{(1+\beta)^2}\left|\dfrac{(1+\beta)^2+2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)}-\mu\right|, & \left|\dfrac{(1+\beta)^2+2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)}-\mu\right|\ge\dfrac{(1+\beta)^2}{4(1-\alpha)(1+2\beta)},\\[2ex] \dfrac{1-\alpha}{1+2\beta}, & \left|\dfrac{(1+\beta)^2+2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)}-\mu\right|\le\dfrac{(1+\beta)^2}{4(1-\alpha)(1+2\beta)}.\end{cases}$$

Proof. Let $f\in M_\beta(\alpha)$, $\beta\ge 0$, $\alpha\in[0,1)$ and $\mu\in\mathbb{C}$. Then,

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}=\alpha+(1-\alpha)p(z), \tag{2.1}$$

where the function $p \in \mathrm{P}$.

As a result of simple simplification from (2.1), we have (see also [13])

$$a_2 = \frac{1-\alpha}{1+\beta} p_1, \tag{2.2}$$

$$a_3 = \frac{1-\alpha}{2(1+2\beta)} p_2 + \frac{(1+3\beta)(1-\alpha)^2}{2(1+\beta)^2(1+2\beta)} p_1^2. \tag{2.3}$$

From (2.2) and (2.3) for $a_3 - \mu a_2^2$ and $\mu \in \mathbb{C}$, we can write

$$a_3 - \mu a_2^2 = \frac{1-\alpha}{2(1+2\beta)} p_2 + \frac{(1-\alpha)^2}{(1+\beta)^2}\left[\frac{(1+3\beta)}{2(1+2\beta)} - \mu\right] p_1^2. \tag{2.4}$$

Using Lemma 1.1, we write the following expression for p_2

$$p_2 = \frac{1}{2}\left[p_1^2 + \left(4-p_1^2\right)x\right] \tag{2.5}$$

for some x with $|x| \le 1$.

Substituting the expression (2.5) in the equality (2.4) and using triangle inequality and then letting $|x| = \xi$ and $|p_1| = t \in [0,2]$ for $\left|a_3 - \mu a_2^2\right|$, we write:

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\alpha)^2}{(1+\beta)^2}\left|\frac{(1+\beta)^2 + 2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)} - \mu\right| t^2 + \frac{(1-\alpha)\left(4-t^2\right)}{4(1+2\beta)}\xi, \ \xi \in [0,1].$$

From this, we can easily write

$$\left|a_3 - \mu a_2^2\right| \le (1-\alpha)\left\{\left[\frac{1-\alpha}{(1+\beta)^2}\left|\frac{(1+\beta)^2 + 2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)} - \mu\right| - \frac{1}{4(1+2\beta)}\right] t^2 + \frac{1}{1+2\beta}\right\}. \tag{2.6}$$

Let define the function $h:[0,2] \to \mathbb{R}$ as follows:

$$h(t) = \left\{\left[\frac{1-\alpha}{(1+\beta)^2}\left|\frac{(1+\beta)^2 + 2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)} - \mu\right| - \frac{1}{4(1+2\beta)}\right] t^2 + \frac{1}{1+2\beta}\right\}. \tag{2.7}$$

We now use elementary calculus to find the maximum of the function $h(t)$.

It is clear that $h'(t) \ge 0$ if

$$\left|\frac{(1+\beta)^2 + 2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)} - \mu\right| \ge \frac{(1+\beta)^2}{4(1-\alpha)(1+2\beta)} \tag{2.8}$$

and $h'(t) \le 0$ otherwise.

Thus, $h(t)$ is increasing function if satisfied the condition (2.8) and is decreasing otherwise. With this completes the proof of Theorem 2.1.

In the special cases, from Theorem 2.1 we arrive at the following results.

Theorem 2.2. *Let the function* $f(z)$ *given by (1.1) be in the class* M_β, $\beta \ge 0$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{4}{(1+\beta)^2}\left|\dfrac{\beta^2+8\beta+3}{4(1+2\beta)}-\mu\right|, & \left|\dfrac{\beta^2+8\beta+3}{4(1+2\beta)}-\mu\right| \ge \dfrac{(1+\beta)^2}{4(1+2\beta)}, \\ \dfrac{1}{1+2\beta}, & \left|\dfrac{\beta^2+8\beta+3}{4(1+2\beta)}-\mu\right| \le \dfrac{(1+\beta)^2}{4(1+2\beta)}. \end{cases}$$

Theorem 2.3. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(\alpha)$, $\alpha \in [0,1)$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 4(1-\alpha)^2\left|\dfrac{3-2\alpha}{4(1-\alpha)}-\mu\right|, & \left|\dfrac{3-2\alpha}{4(1-\alpha)}-\mu\right| \ge \dfrac{1}{4(1-\alpha)}, \\ 1-\alpha, & \left|\dfrac{3-2\alpha}{4(1-\alpha)}-\mu\right| \le \dfrac{1}{4(1-\alpha)}. \end{cases}$$

Theorem 2.4. *Let the function* $f(z)$ *given by (1.1) be in the class* $C(\alpha)$, $\alpha \in [0,1)$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} (1-\alpha)^2\left|\dfrac{3-2\alpha}{3(1-\alpha)}-\mu\right|, & \left|\dfrac{3-2\alpha}{3(1-\alpha)}-\mu\right| \ge \dfrac{1}{3(1-\alpha)}, \\ \dfrac{1-\alpha}{3}, & \left|\dfrac{3-2\alpha}{3(1-\alpha)}-\mu\right| \le \dfrac{1}{3(1-\alpha)}. \end{cases}$$

Corollary 2.1. *Let the function* $f(z)$ *given by (1.1) be in the class* S^* *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} |3-4\mu|, & |3-4\mu| \ge 1, \\ 1, & |3-4\mu| \le 1. \end{cases}$$

Corollary 2.2. *Let the function* $f(z)$ *given by (1.1) be in the class* C *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} |1-\mu|, & |1-\mu| \ge \dfrac{1}{3}, \\ \dfrac{1}{3}, & |1-\mu| \le \dfrac{1}{3}. \end{cases}$$

We give the following theorem which will prove in similar to the proof of Theorem 2.1.

Theorem 2.5. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\beta(\alpha), \beta \ge 0$, $\alpha \in [0,1)$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{4(1-\alpha)^2}{(1+\beta)^2}\left|\dfrac{(1+\beta)^2+2(1-\alpha)(1+3\beta)}{4(1-\alpha)(1+2\beta)}-\mu\right|, & \mu \le \mu_0 \text{ or } \mu \ge \mu_1, \\ \dfrac{1-\alpha}{1+2\beta}, & \mu_0 \le \mu \le \mu_1, \end{cases}$$

where $\mu_0 = \dfrac{(1-\alpha)(1+3\beta)}{2(1-\alpha)(1+2\beta)}$ *and* $\mu_1 = \dfrac{(1+\beta)^2+(1-\alpha)(1+3\beta)}{2(1-\alpha)(1+2\beta)}$.

From the Theorem 2.5, we arrive at the following results.

Theorem 2.6. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\beta, \beta \ge 0$, *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \frac{4}{(1+\beta)^2}\begin{cases}\left|\dfrac{\beta^2+10\beta+3}{4(1+2\beta)}-\mu\right|, & \mu\le\mu_0 \text{ or } \mu\ge\mu_1,\\ \dfrac{(1+\beta)^2}{4(1+2\beta)}, & \mu_0\le\mu\le\mu_1,\end{cases}$$

where $\mu_0=\dfrac{1+3\beta}{2(1+2\beta)}$ *and* $\mu_1=\dfrac{\beta^2+5\beta+2}{2(1+2\beta)}$.

Theorem 2.7. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(\alpha)$, $\alpha\in[0,1)$ *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le 4(1-\alpha)^2\begin{cases}\left|\dfrac{3-2\alpha}{4(1-\alpha)}-\mu\right|, & \mu\le\mu_0 \text{ or } \mu\ge\mu_1,\\ \dfrac{1}{4(1-\alpha)}, & \mu_0\le\mu\le\mu_1,\end{cases}$$

where $\mu_0=\dfrac{1}{2}$ *and* $\mu_1=\dfrac{2-\alpha}{2(1-\alpha)}$

Theorem 2.8. *Let the function* $f(z)$ *given by (1.1) be in the class* $C(\alpha)$, $\alpha\in[0,1)$ *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le (1-\alpha)^2\begin{cases}\left|\dfrac{3-2\alpha}{3(1-\alpha)}-\mu\right|, & \mu\le\mu_0 \text{ or } \mu\ge\mu_1,\\ \dfrac{1}{3(1-\alpha)}, & \mu_0\le\mu\le\mu_1,\end{cases}$$

where $\mu_0=\dfrac{2}{3}$ *and* $\mu_1=\dfrac{2(2-\alpha)}{3(1-\alpha)}$.

Corollary 2.3. *Let the function* $f(z)$ *given by (1.1) be in the class* S^* *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases}|3-4\mu|, & \mu\le\dfrac{1}{2} \text{ or } \mu\ge 1,\\ 1, & \dfrac{1}{2}\le\mu\le 1.\end{cases}$$

Corollary 2.4. *Let the function* $f(z)$ *given by (1.1) be in the class* C *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases}|1-\mu|, & \mu\le\dfrac{2}{3} \text{ or } \mu\ge\dfrac{4}{3},\\ \dfrac{1}{3}, & \dfrac{2}{3}\le\mu\le\dfrac{4}{3}.\end{cases}$$

Taking $\mu=0$ in Theorem 2.5, we obtain the following inequality for $|a_3|$, which verifies result obtained in [13].

Corollary 2.5. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\beta(\alpha)$, $\beta\ge 0$, $\alpha\in[0,1)$. *Then,*

$$|a_3| \le \frac{1-\alpha}{1+2\beta}\left[1+\frac{2(1+3\beta)(1-\alpha)}{(1+\beta)^2}\right].$$

Taking $\mu = 1$ in Theorem 2.5, we arrive at the following results for the Fekete-Szegö functional $|a_3 - a_2^2|$.

Corollary 2.6. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\beta(\alpha), \beta \ge 0$, $\alpha \in [0,1)$. *Then,*

$$|a_3 - a_2^2| \le \frac{1-\alpha}{1+2\beta}.$$

Corollary 2.7. *Let the function* $f(z)$ *given by (1.1) be in the class* $M_\beta, \beta \ge 0$. *Then,*

$$|a_3 - a_2^2| \le \frac{1}{1+2\beta}.$$

Corollary 2.8. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(\alpha)$, $\alpha \in [0,1)$. *Then,*

$$|a_3 - a_2^2| \le 1-\alpha.$$

Corollary 2.9. *Let the function* $f(z)$ *given by (1.1) be in the class* $C(\alpha)$, $\alpha \in [0,1)$. *Then,*

$$|a_3 - a_2^2| \le \frac{1-\alpha}{3}.$$

Corollary 2.10. *Let the function* $f(z)$ *given by (1.1) be in the class* S^*. *Then,*

$$|a_3 - a_2^2| \le 1.$$

Corollary 2.11. *Let the function* $f(z)$ *given by (1.1) be in the class* C. *Then,*

$$|a_3 - a_2^2| \le \frac{1}{3}.$$

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