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The Squares in Euler Quotient

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Abstract For any positive integer n, let $\varphi(n)$ and $\omega(n)$ denote the Euler function and the number of distinct prime divisors of n respectively. Further, for any positive a such that a>1 and gcd(a,n)=1, the positive integer of the form $(a^{\varphi(n)}-1)/n$ is called an Euler quotient. In this paper, using some elementary number theory methods, the squares in Euler quotients are discussed and all squares with $\omega(n)\le 2$ are determined.

Keywords Euler function; Euler quotient; squares

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1. Introduction

Let denote the set of all positive integers. For positive integer n, let $\varphi(n)$ and $\omega(n)$ denote the Euler function of n and the number of distinct prime factors of n. According to the famous Euler theorem of number theory: when a and n coprime and a is an integer, there must be $a^{\varphi(n)}\equiv 1(\mathrm{mod}\, n)$ (following the theorem 2.3.2 as used in [1]). Therefore, if a>1 and a is an integer relatively prime with n, $(a^{\varphi(n)}-1)/n$ must be positive integer, which is called Euler quotient. When $n=p$, p is prime number and $\varphi(p)=p-1$ is given, thus in the condition of a>1 and a is not divisible by p , $\left(a^{p-1}-1\right)/p$ is also positive integer, which is called Fermat quotient. Obviously, Fermat quotient is an exceptional case of Euler quotient. Among the discussions of number theory including Fermat conjecture, the arithmetic property of Fermat quotient and Euler quotient is a remarkable research subject (following the arguments as used in [1] and [2]).

This paper discusses the squares in Euler quotient. According to the definition of Euler quotient, it can be stated as the solution of following equation

$$x^{\varphi(n)}-1=ny^2,\quad x,y\in \qquad (1.1)$$

For that, [3] figures out the situation when n is greater than 3 and n is odd prime number, this paper works out the situation when $\omega(n)\le 2$ further, it proves:

Theorem When $\omega(n)\le 2$, equation (1.1) only has following solution:

(i) $(n,x,y)=(1,t^2+1,t)$, where t is positive integer.

(ii) $(n,x,y)=(2,2t^2+1,t)$, where t is positive integer.

(iii) $(n,x,y)=(3,u_k,v_k)$, where u_k and v_k satisfy $u_k+v_k\sqrt{3}=\left(2+\sqrt{3}\right)^k$, k is positive integer.

(iv) $(n,x,y)=(5,3,4)$.

(v) $(n,x,y)=(6,u_k',v_k')$, where u_k' and v_k' satisfy $u_k'+v_k'\sqrt{6}=(5+2\sqrt{6})^k$, k is positive integer.

(vi) $(n,x,y)=(7,2,3)$.

From the theorem above, it implies that when $\omega(n)\le 2$,the equation(1.1) has no solution (x,y) if $n>7$. Therefore, this paper presents the following conjecture:

Conjecture When $n>7$, equation(1.1) has no solution (x,y).

2. Lemmas

Lemma 2.1 if $n=p_1^{r_1}\cdots p_k^{r_k}$ is canonical decomposition of positive n, then $\varphi(n)=p_1^{r-1_1}\cdots p_k^{r_k-1}(p_1-1)\cdots(p_k-1)$.

Demonstration Following the theorem 2.5.4 as used in [1].

Lemma 2.2 when $n>2$, $\varphi(n)$ must be even number.

Demonstration According to the definition of Euler function, $\varphi(n)$ is equal to the number of positive integer which is not greater than n and coprime with n. When $n>2$, if positive integer a satisfies $1\le a\le n$ and $\gcd(a,n)=1$, there must be $1\le n-a\le n$ and $\gcd(n-a,n)=1$. Because $n/2$ is not integer or coprime with n, thus $a\ne n-a$. Hence the number of positive integer which is not greater than n and coprime with n is even number.This completes the proof.

Lemma 2.3 When n >1 and n is square, equation (1.1) has no solution (x,y).

Demonstration when n >1 and n is square, $n\ge 4$, then lemma 2.2 implies that $\varphi(n)$ is even number. Therefore, $1=x^{\varphi(n)}-ny^2=\left(x^{\varphi(n)/2}\right)^2-\left(\sqrt{n}y\right)^2\ge x^{\varphi(n)/2}+\sqrt{n}y>1$ is contradictory from (1.1), which is not possible. This completes the proof.

Lemma 2.4 Let D denote non-square positive integer. Equation

$$u^2-Dv^2=1,u,v\in\square \tag{2.1}$$

must has solution (u,v), and it has the only one solution (u_1,v_1) which satisfies $u_1+v_1\sqrt{D}\le u+v\sqrt{D}$, where (u,v) is all solutions of this equation. Then (u_1,v_1) is the minimal solution of equation (2.1). Here, $(u,v)=(u_k,v_k)(k=1,2,\cdots)$ is all solutions of equation (2.1), where $u_k+v_k\sqrt{D}=(u_1+v_1\sqrt{D})^k$.

Demonstration Following the theorem 10.9.1 and 10.9.2 as used in [1].

Lemma 2.5 When $D\in\{1,2,3,6\}$, equation

$$X^3-1=DY^2,\quad X,Y\in\square \tag{2.2}$$

has no solution (X,Y); equation

$$X^3+1=DY^2,\quad X,Y\in\square \tag{2.3}$$

only has solution $(D,X,Y)=(1,2,3)$, (2,1,1) and (2,23,78).

Demonstration Following the theorem 6.2.5 as used in [4].

Lemma 2.6 For prime number p, equation

$$X^4-pY^2=1,\ X,Y\in\square \tag{2.4}$$

only has solution $(p,X,Y)=(5,3,4)$ and (29,99,1820).

Demonstration Following the arguments as used in [5].

Lemma 2.7 For odd prime number p, equation

$$X^4 - 2pY^2 = 1,\ X, Y \in \square \tag{2.5}$$

only has solution $(p, X, Y) = (3,7,20)$.

Demonstration Following the arguments as used in [6].

Lemma 2.8 Equation

$$X^2 \pm 1 = Y^m,\ X, Y, m \in \square \quad , m > 1 \tag{2.6}$$

Only has solution $(X, Y, m) = (3,2,3)$.

Demonstration Following the arguments as used in [7] and [8].

Lemma 2.9 Equation

$$X^m \pm 1 = 2Y^2,\ X, Y, m \in \square \quad , X > 1,\ m \geq 4 \tag{2.7}$$

only has solution $(X, Y, m) = (3,11,5)$.

Demonstration Following the arguments as used in [9].

Lemma 2.10 When p is greater than 3 and p is odd prime number, equation

$$X^{p-1} - 1 = pY^2,\ X, Y \in \square \tag{2.8}$$

only has solution $(p, X, Y) = (5,3,4)$ and (7,2,3); equation

$$X^{p-1} - 1 = 2pY^2,\ X, Y \in \square \tag{2.9}$$

has no solution (X, Y).

Demonstration Following the arguments as used in [3].

3. Demonstration of Theorem

Because $\varphi(1) = \varphi(2) = 1$ is known from lemma 2.1, thus equation (1.1) only has solution (i) and (ii) of this lemma respectively when $n = 1$ and 2.

When $n = 3$, because $\varphi(3) = 2$, thus (1.1) deduces $x^2 - 3y^2 = 1$. Hence equation (1.1) only has solution $(X, Y) = (u, v)$ here, where (u, v) is the solution of equation (2.1) when $D = 3$. Since the minimal solution of equation (2.1) is $(u_1, v_1) = (2,1)$ when $D = 3$, so equation(1.1) only has solution (iii) here in accordance with lemma 2.4. Likewise, when $n = 6$, because $\varphi(6) = 2$ and equation(2.1) has the minimal solution $(u_1, v_1) = (5,2)$ when $D = 6$, thus equation(1.1) only has solution (v) here.

When $n = 4$ and 9, lemma 2.3 deduces that equation (1.1) has no solution. When $n = 5$ and 7, lemma 2.10 deduces that equation (1.1) has solution (iv) and (vi) respectively.

It can be seen from the analysis above: if equation (1.1) has no solution can be proved when $w(n) \leq 2$, $n = 8$ or $n \geq 10$, then this lemma holds. Let (x, y) denote a set of solution of equation (1.1), thus $x > 1$ because of $ny^2 > 0$. The solution will be proved does not exist through the following four conditions.

Condition Ⅰ $n = 2^r$, where r is greater than 2 and r is positive integer.

From lemma 2.1, $\varphi(2^r) = 2^{r-1}$ is known. Because $r - 1 \geq 2$, then (1.1) deduces

$$\left(x^{2^{r-3}}\right)^4 - 1 = 2^r y^2,\ x > 1. \tag{3.1}$$

However, according to lemma 2.6 and 2.8, (3.1) doesn't hold.

Condition Ⅱ $n = p^s$,where p is odd prime number, s is positive integer.

Because $\varphi(p^s) = p^{s-1}(p-1)$, then (1.1) deduces

$$\left(x^{p^{s-1}}\right)^{p-1}-1=p^s y^2\ ,\ x>1\ . \tag{3.2}$$

On account of $p^s>8$, (3.2) doesn't hold when $p>3$ according to lemma 2.8 and 2.10.

When $p=3$, because $3^s>8$, thus $s\geq 2$, and (3.2) deduces

$$\left(x^{2\cdot 3^{s-2}}\right)^3-1=3^s y^2\ ,\ x>1\ . \tag{3.3}$$

However, according to lemma 2.5 and 2.8, (3.3) doesn't hold.

Condition Ⅲ $n=2^r p^s$, where p is odd prime number, r and s are positive integer.

When $r=1$ and s is even number, then (1.1) deduces

$$\left(x^{p^{s-2}}\right)^{(p-1)p}-1=2(p^{s/2}y)^2\ ,\ x>1\ . \tag{3.4}$$

Whereas, because $(p-1)p$ is even number and greater than 4, then lemma 2.9 implies that (3.4) doesn't hold.

When $r=1$ and s is odd prime number, (1.1) deduces

$$\left(x^{p^{s-1}}\right)^{p-1}-1=2p(p^{(s-1)/2}y)^2\ ,\ x>1 \tag{3.5}$$

According to lemma 2.10, $p=3$ and $s>1$ are known from (3.5). Here, (3.5) deduces

$$\left(x^{2\cdot 3^{s-2}}\right)^3-1=2\cdot 3^s y^2\ ,\ x>1\ . \tag{3.6}$$

However, （3.6）doesn't hold in accordance with lemma 2.5.

When $r>1$, (1.1) deduces

$$\left(x^{2^{r-2}p^{s-1}(p-1)/2}\right)^4-1=2^r p^s y^2\ ,\ x>1. \tag{3.7}$$

However, (3.7) is not established is known from lemma 2.6 and 2.7. Hence, equation (1.1) has no solution here.

Condition Ⅳ $n=p^r q^s$, where p and q are distinct odd prime number, r and s are positive integer.

Here lemma 2.1 deduces

$$\varphi\left(p^r q^s\right)=p^{r-1}q^{s-1}(p-1)(q-1)\ . \tag{3.8}$$

(3.8) deduces that $\varphi\left(p^r q^s\right)$ is the multiple of 4, then

$$\varphi\left(p^r q^s\right)=4f\quad f\in \tag{3.9}$$

Because $p\neq q$, thus $\varphi\left(p^r q^s\right)>4$, $f>1$ is known from （3.9）. Then (1.1) and (3.9) deduce

$$x^{4f}-1=p^r q^s y^2\ ,\ x>1. \tag{3.10}$$

Because $x^{4f}-1=\left(x^{2f}-1\right)\left(x^{2f}+1\right)$, where $x^{2f}-1$ and $x^{2f}+1$ satisfy the positive integer of

$$\gcd(x^{2f}-1,x^{2f}+1)=\begin{cases}1,\text{如果2丨x},\\ 2,\text{如果2不x},\end{cases} \tag{3.11}$$

Moreover, due to $2f\geq 4$, then lemma 2.8 and 2.9 deduce

$$x^{2f}\pm 1\neq z^2\ \text{或}\ 2z^2,\quad z\in\ , \tag{3.12}$$

Thus (3.10), (3.11) and (3.12) deduce

$$x^{2f}-1=\begin{cases}p^r a^2,\\ 2p^r a^2,\end{cases}\quad x^{2f}+1=\begin{cases}q^s b^2,\\ 2q^s b^2,\end{cases}=,\quad y=\begin{cases}ab,\text{如果2丨x},\\ 2ab,\text{如果2不x},\end{cases}$$

$$a,b\in\ ,\ \gcd(a,b)=1 \tag{3.13}$$

From the second equality of (3.13), $q\equiv 1(\mathrm{mod}\,4)$ is known, thus f is even number from (3.8) and (3.9), that is, $f=2g$, where g is positive integer. However, (3.13) doesn't hold here according to lemma 2.6 and 2.7.

In conclusion: when $\omega(n) \leq 2$, if $n = 8$ or $n \geq 10$, equation(1.1) has no solution (x, y). This completes the proof.

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