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Research Article

Application of the second kind Chebyshev polynomials to the Fekete-Szegö problem of certain class analytic functions

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Abstract In this work, making the second kind Chebyshev polynomials, we introduce and investigate new subclasses of analytic functions on the open unit disk in the complex plane. We give upper bound estimates for the Fekete-Szegö functional of the function belonging to these classes.

Keywords Analytic Function, Univalent Function, Coefficient Bound, Chebyshev Polynomials
AMS Subject Classification: 30A10, 30C45, 30C50, 30C55

1. Introduction

Let A be the class of all complex valued analytic functions on $U=\{z\in\mathbb{C}:\ |z|<1\}$ $f(z)$ given by the following series

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}, \tag{1.1}$$

normalized by $f(0)=0=f'(0)-1$.

Also, let S be the class of all functions in A which are univalent in U. Some of the important and well-known subclasses of S are well known classes S^* and C given below (see [3, 4, 10])

$$S^*=\left\{f\in S:\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in U\right\}$$ and $$C=\left\{f\in S:\ \operatorname{Re}\left(\frac{(zf'(z))'}{f'(z)}\right)>0,\ z\in U\right\}$$

such that is said starlike and convex functions in U, respectively.

It is well known that, an analytic function f is subordinate to an analytic function ϕ and written $f(z)\prec\phi(z)$, provided that there is an analytic function $\omega(z)$ defined on U with $\omega(0)=0$ and $|\omega(z)|<1$ satisfying $f(z)=\phi(\omega(z))$. Ma and Minda [7] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $\frac{(zf'(z))'}{f'(z)}=1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general function. For this purpose, they considered an analytic function $\phi(z)$ with positive real part in U, $\phi(0)=1$, $\phi'(0)>0$. The class of Ma-Minda starlike and Ma-Minda convex functions consists of the functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$, respectively.

It is well known that Chebyshev polynomials play a considerable act in numerical and mathematical physics. Also, it is well known that Chebyshev polynomials are four kinds. The well-known kinds of the Chebyshev polynomials are the first and second kinds. The most of research articles related to specific orthogonal polynomials of Chebyshev family, contain essentially results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$, and their numerous uses in different applications (see [2]).

In this paper, we will use the second kind Chebyshev polynomials to investigation of the coefficient and Fekete-Szegö problems of certain subclass of analytic functions.

It is well known that in the case of real variable x on the interval $(-1,1)$, the second kind Chebyshev polynomial is defined by

$$U_n(x)=\frac{\sin\left[(n+1)\arccos x\right]}{\sin(\arccos x)}=\frac{\sin\left[(n+1)\arccos x\right]}{\sqrt{1-x^2}}. \tag{1.2}$$

It follows from (1.2) that

$$U_{n+1}(t)=2tU_n(t)-U_{n-1}(t),\ n=1,2,3,... \tag{1.3}$$

Since $U_0(t)=1,\ U_1(t)=2t$, from (1.3) we obtain $U_2(t)=4t^2-1$ and $U_3(t)=4t\left(2t^2-1\right)$.

We consider the function

$$G(t,z)=\frac{1}{1-2tz+z^2},\ t\in(0,1),\ z\in U.$$

It is well known that

$$G(t,z)=1+2\cos\alpha\cdot z+\left(3\cos^2\alpha-\sin^2\alpha\right)z^2+\left(8\cos^3\alpha-4\cos\alpha\right)z^3+\cdots,\ z\in U$$

for $t=\cos\alpha,\ \alpha\in\left(0,\frac{\pi}{2}\right)$. Thus, the function $G(t,z)$, $t\in(0,1)$, $z\in U$ is analytic respect to parameter z in U and has the following series expansion

$$G(t,z)=1+U_1(t)z+U_2(t)z^2+U_3(t)z^3+\cdots,\quad z\in U, \tag{1.4}$$

where $U_n(t),\ n=1,2,3,...$ is second kind Chebyshev polynomial.

It can be easily verified that $G(t,0)=1$ and $G_z'(t,0)>0$.

So that, using the function $G(t,z)$ we define a subclass of univalent functions as follows.

Definition 1.1. *A function* $f \in S$ *given by (1.1) is said to be in the class* $S^*C(G;\beta,t)$, $\beta \geq 0$, $t \in (0,1)$, *where* G *is the function given by (1.4), if the following condition is satisfied*

$$\frac{zf'(z)+\beta z^2 f''(z)}{\beta zf'(z)+(1-\beta)f(z)} \prec G(t,z),\ z \in U.$$

Definition 1.2. *A function* $f \in S$ *given by (1.1) is said to be in the class* $S^*(G;t)$, $t \in (0,1)$, *where* G *is the function given by (1.4), if the following condition is satisfied*

$$\frac{zf'(z)}{f(z)} \prec G(t,z),\ z \in U.$$

Definition 1.3. *A function* $f \in S$ *given by (1.1) is said to be in the class* $C(G;t)$, $t \in (0,1)$, *where* G *is the function given by (1.4), if the following condition is satisfied*

$$1+\frac{zf''(z)}{f'(z)} \prec G(t,z),\ z \in U.$$

It is well known that, one of the important tools in the theory of analytic functions is the functional $H_2(1)=a_3-a_2^2$ which is known the Fekete-Szegö functional and one usually considers the further generalized functional $a_3-\mu a_2^2$, where is a (real or complex) number (see [11]). Estimating the upper bound estimate of $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö problem. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions (see [1, 5, 8, 9, 11]).

In this paper, making use of the second kind Chebyshev polynomial, we introduce and investigate new subclasses $S^*C(G;\beta,t)$, $\beta \geq 0$, $S^*C(G;t)$ and $C(G;t)$ for $t \in (0,1)$ of the analytic functions on the open unit disk U. Here, we solve Fekete-Szegö problem for these subclasses.

To prove our main results, we shall require the following well known lemma.

Lemma 1.1. ([3]) *Let* P *be the class of all analytic functions* $p(z)$ *of the form*

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots=1+\sum_{n=1}^{\infty}p_nz^n$$

with $\operatorname{Re} p(z)>0$, *for* $z \in U$ *and* $p(0)=1$. *Then,* $|p_n| \leq 2$ *for every* $n=1,2,3,\ldots$. *These inequalities are sharp for each* $n=1,2,3,\ldots$. *Moreover,*

$$2p_2=p_1^2+\left(4-p_1^2\right)x,$$

$$4p_3=p_1^3+2\left(4-p_1^2\right)p_1x-\left(4-p_1^2\right)p_1x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z$$

for some complex $x,\ z$ *with* $|x| \leq 1$, $|z| \leq 1$.

2. Fekete-Szegö problem

In this section, we investigate Fekete-Szegö problem for the function classes defined in the first section.

Firstly we will prove the following theorem on upper bound of the Fekete-Szegö functional for the function belonging to the class $S^*C(G;\beta,t)$.

Theorem 2.1. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*C(G;\beta,t)$, $\beta \geq 0$, $t \in (0,1)$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \left(\frac{2t}{1+\beta}\right)^2 \left|\frac{(1+\beta)^2(8t^2-1)}{8(1+2\beta)t^2} - \mu\right| & \text{if } \frac{(1+\beta)^2}{4(1+2\beta)t} \leq \left|\frac{(1+\beta)^2(8t^2-1)}{8(1+2\beta)t^2} - \mu\right|, \\ \frac{t}{1+2\beta} & \text{if } \frac{(1+\beta)^2}{4(1+2\beta)t} \geq \left|\frac{(1+\beta)^2(8t^2-1)}{8(1+2\beta)t^2} - \mu\right|. \end{cases}$$

Proof. Let $f \in S^*C(G;\beta,t)$, $\beta \geq 0$, $t \in (0,1)$; that is,

$$\frac{zf'(z) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta) f(z)} \prec G(t,z),\ z \in U. \tag{2.1}$$

Define the function $F : U \to \mathbb{C}$ by

$$F(z) = \beta z f'(z) + (1-\beta) f(z),\ z \in U,$$

the condition (2.1) we can write as follows

$$\frac{zF'(z)}{F(z)} \prec G(t,z),\ z \in U.$$

It is clear that

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n,\ z \in U,$$

where $A_n = (1+(n-1)\beta) a_n$, $n = 2,3,4,\ldots$.

So that, the function $F(z)$ belong to the class $S^*(G;t)$, $t \in (0,1)$. Then, there is an analytic function $\omega : U \to U$ with $\omega(0) = 0$, $|\omega(z)| < 1$ satisfying the following condition

$$\frac{zF'(z)}{F(z)} = G(t,\omega(z)),\ z \in U. \tag{2.2}$$

Let the function $p \in \mathrm{P}$ be defined as follows

$$p(z):=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+\cdots$$

It follows that

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)z^3+\cdots\right]. \tag{2.3}$$

Taking $z\equiv\omega(z)$ in (1.4), we get

$$G(t,\omega(z))=1+\frac{U_1(t)}{2}p_1z+\left[\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2\right]z^2$$
$$+\left[\frac{U_1(t)}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{U_2(t)}{2}p_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_3(t)}{8}p_1^3\right]+\cdots. \tag{2.4}$$

Substituting this expression of the function $G(t,\omega(z))$ in (2.2) with sample computation, we write

$$z+2A_2z^2+3A_3z^3+4A_4z^4+\cdots=z+\left[A_2+\frac{U_1(t)}{2}p_1\right]z^2$$
$$+\left[A_3+A_2\frac{U_1(t)}{2}p_1+\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2\right]z^3$$
$$+\left\{\begin{array}{l}A_4+A_3\dfrac{U_1(t)}{2}p_1+A_2\left[\dfrac{U_1(t)}{2}\left(p_2-\dfrac{p_1^2}{2}\right)+\dfrac{U_2(t)}{4}p_1^2\right]\\+\dfrac{U_1(t)}{2}\left(p_3-p_1p_2+\dfrac{p_1^3}{4}\right)+\dfrac{U_2(t)}{2}p_1\left(p_2-\dfrac{p_1^2}{2}\right)+\dfrac{U_3(t)}{8}p_1^3\end{array}\right\}z^4.$$

Comparing the coefficients of the like power of z in both sides of the last equality with sample computation, we get

$$A_2=\frac{U_1(t)}{2}p_1, \tag{2.5}$$

$$A_3=\frac{U_1^2(t)}{8}p_1^2+\frac{U_1(t)}{4}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{8}p_1^2, \tag{2.6}$$

$$A_4 = \frac{U_1(t)}{6} p_1 A_3 + \frac{A_2}{3}\left[\frac{U_1(t)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_2(t)}{4} p_1^2\right]$$
$$+\frac{U_1(t)}{6}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{U_2(t)}{6} p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_3(t)}{24} p_1^3. \tag{2.7}$$

Using equalities $A_n = (1+(n-1)\beta)a_n$, $n = 2,3,4$ from the equalities (2.5)-(2.7) we obtain the expression for a_2, a_3 and a_4 as follows

$$a_2 = \frac{U_1(t)}{2(1+\beta)} p_1, \tag{2.8}$$

$$a_3 = \frac{1}{1+2\beta}\left[\frac{U_1^2(t)+U_2(t)}{8} p_1^2 + \frac{U_1(t)}{4}\left(p_2 - \frac{p_1^2}{2}\right)\right], \tag{2.9}$$

$$a_4 = \frac{1}{1+3\beta}\left\{\frac{(1+2\beta)U_1(t)}{6} p_1 a_3 + \frac{1+\beta}{3} a_2\left[\frac{U_1(t)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_2(t)}{4} p_1^2\right]\right.$$
$$\left.+\frac{U_1(t)}{6}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{U_2(t)}{6} p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_3(t)}{24} p_1^3\right\}. \tag{2.10}$$

From the equalities (2.8) and (2.9) for $a_3 - \mu a_2^2$ we can write

$$a_3 - \mu a_2^2 = \frac{U_1^2(t)}{4(1+\beta)^2}\left\{\frac{(1+\beta)^2\left[U_1^2(t)+U_2(t)\right]}{2(1+2\beta)U_1^2(t)} - \mu\right\} p_1^2 + \frac{U_1(t)}{4(1+2\beta)}\left(p_2 - \frac{p_1^2}{2}\right). \tag{2.11}$$

From the Lemma 1.1, we write

$$p_2 - \frac{p_1^2}{2} = \frac{(4-p_1^2)x}{2} \tag{2.12}$$

for some x with $|x| \le 1$.

Substituting the expression (2.12) in the equality (2.11) and using triangle inequality, letting $|x| = \xi$ and $|p_1| = t$ for the $|a_3 - \mu a_2^2|$ we obtain

$$|a_3 - \mu a_2^2| \le c_1(t,\tau)\xi + c_2(t,\tau), \tag{2.13}$$

where

$$c_1(t,\tau)=\frac{U_1(t)(4-\tau^2)}{8(1+2\beta)},\quad c_2(t,\tau)=\frac{U_1^2(t)}{4(1+\beta)^2}\left|\frac{(1+\beta)^2\left[U_1^2(t)+U_2(t)\right]}{2(1+2\beta)U_1^2(t)}-\mu\right|\tau^2,$$

$$\xi\in[0,1],\ \tau\in[0,2],\ t\in(0,1).$$

Since $c_1(t,\tau)\geq 0$ for all $\tau\in[0,2]$ and $t\in(0,1)$, from the inequality (2.13) for the $\left|a_3-\mu a_2^2\right|$ we get

$$\left|a_3-\mu a_2^2\right|\leq c_1(t,\tau)+c_2(t,\tau);$$

that is,

$$\left|a_3-\mu a_2^2\right|\leq a(t,\beta)\tau^2+b(t,\beta), \tag{2.14}$$

where

$$a(t,\beta)=\frac{U_1(t)}{4}\left\{\frac{U_1(t)}{(1+\beta)^2}\left|\frac{(1+\beta)^2\left[U_1^2(t)+U_2(t)\right]}{2(1+2\beta)U_1^2(t)}-\mu\right|-\frac{1}{2(1+2\beta)}\right\},\quad b(t,\beta)=\frac{U_1(t)}{2(1+2\beta)}.$$

We define the function $\varphi:[0,2]\to\mathbb{R}$ as follows:

$$\varphi(\tau)=a(t,\beta)\tau^2+b(t,\beta),\ \tau\in[0,2] \tag{2.15}$$

for fixed values of $t\in(0,1)$ and $\beta\geq 0$.

Thus, the function $\varphi(\tau)$ is increasing function if

$$\frac{(1+\beta)^2}{2(1+2\beta)U_1(t)}\leq\left|\frac{(1+\beta)^2\left[U_1^2(t)+U_2(t)\right]}{2(1+2\beta)U_1^2(t)}-\mu\right| \tag{2.16}$$

and is decreasing function if

$$\left|\frac{(1+\beta)^2\left[U_1^2(t)+U_2(t)\right]}{2(1+2\beta)U_1^2(t)}-\mu\right|\leq\frac{(1+\beta)^2}{2(1+2\beta)U_1(t)} \tag{2.17}$$

on the interval $[0,2]$.

So that, from (2.14) and (2.15), we obtain

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{U_1^2(t)}{(1+\beta)^2}\left|\dfrac{(1+\beta)^2\left[U_1^2(t)+U_2(t)\right]}{2(1+2\beta)U_1^2(t)}-\mu\right| & \text{if satisfied condition (2.16)}, \\ \dfrac{U_1(t)}{2(1+2\beta)} & \text{if satisfied condition (2.17)}. \end{cases}$$

Thus, the proof of Theorem 2.1 is completed.

The following theorems are direct results of the Theorem 2.1.

Theorem 2.2. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(G;t)$, $t\in(0,1)$ *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{t^2}{2}\left|8-t^{-2}-8\mu\right| & \text{if } \dfrac{1}{4t}\le\left|1-\dfrac{1}{8t^2}-\mu\right|, \\ t & \text{if } \dfrac{1}{4t}\ge\left|1-\dfrac{1}{8t^2}-\mu\right|. \end{cases}$$

Theorem 2.3 *Let the function* $f(z)$ *given by (1.1) be in the class* $C(G;t)$, $t\in(0,1)$ *and* $\mu\in\mathbb{C}$. *Then,*

$$\left|a_3-\mu a_2^2\right| \le \frac{t}{6}\cdot\begin{cases} t\left|8-t^{-2}-6\mu\right| & \text{if } \dfrac{2}{t}\le\left|8-t^{-2}-6\mu\right|, \\ 2 & \text{if } \dfrac{2}{t}\ge\left|8-t^{-2}-6\mu\right|. \end{cases}$$

From the obtained above results we arrive at the following results for $|a_3|$.

Corollary 2.1. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*C(G;\beta,t)$, $\beta\ge 0$, $t\in(0,1)$. *Then,*

$$|a_3| \le \frac{1}{2(1+2\beta)}\begin{cases} 1-8t^2, & \text{if } t\in\left(0,\dfrac{1}{4}\right], \\ 2t, & \text{if } t\in\left[\dfrac{1}{4},\dfrac{1}{2}\right], \\ 8t^2-1, & \text{if } t\in\left[\dfrac{1}{4},1\right). \end{cases}$$

Corollary 2.2. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(G;t)$, $t\in(0,1)$. *Then,*

$$|a_3| \le \frac{1}{2}\begin{cases} 1-8t^2, & \text{if } t \in \left(0, \frac{1}{4}\right], \\ 2t, & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ 8t^2-1, & \text{if } t \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Corollary 2.3. *Let the function* $f(z)$ *given by (1.1) be in the class* $C(G;t)$, $t \in (0,1)$. *Then,*

$$|a_3| \le \frac{1}{6}\begin{cases} 1-8t^2, & \text{if } t \in \left(0, \frac{1}{4}\right], \\ 2t, & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ 8t^2-1, & \text{if } t \in \left[\frac{1}{2}, 1\right). \end{cases}$$

The following theorem is direct result of the Theorem 2.1.

Theorem 2.4. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*C(G;\beta,t)$, $\beta \ge 0$, $t \in (0,1)$. *Then,*

$$|a_3 - a_2^2| \le \begin{cases} \left(\frac{2t}{1+\beta}\right)^2 \left|\frac{(1+\beta)^2(8t^2-1)}{8(1+2\beta)t^2} - 1\right| & \text{if } \frac{(1+\beta)^2}{4(1+2\beta)t} \le \left|\frac{(1+\beta)^2(8t^2-1)}{8(1+2\beta)t^2} - 1\right|, \\ \frac{t}{1+2\beta} & \text{if } \frac{(1+\beta)^2}{4(1+2\beta)t} \ge \left|\frac{(1+\beta)^2(8t^2-1)}{8(1+2\beta)t^2} - 1\right|. \end{cases}$$

From the Theorem 2.4 we arrive at the following results for $|a_3 - a_2^2|$.

Corollary 2.4. *Let the function* $f(z)$ *given by (1.1) be in the class* $S^*(G;t)$, $t \in (0,1)$. *Then,*

$$|a_3 - a_2^2| \le \frac{1}{2}\begin{cases} 1, & \text{if } t \in \left(0, \frac{1}{2}\right], \\ 2t, & \text{if } t \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Corollary 2.5. *Let the function* $f(z)$ *given by (1.1) be in the class* $C(G;t)$, $t \in (0,1)$. *Then,*

$$\left|a_3 - a_2^2\right| \le \frac{1}{6}\begin{cases} 1-2t^2, \text{ if } t \in \left(0, \dfrac{\sqrt{3}-1}{2}\right], \\ 2t, \qquad \text{if } t \in \left[\dfrac{\sqrt{3}-1}{2}, 1\right). \end{cases}$$

3. Conclusions

In this work, making the second kind Chebyshev polynomials, was introduced and investigated new subclasses of analytic functions on the open unit disk in the complex plane. Here, was given upper bound estimates for the Fekete-Szegö functional of the function belonging to these classes.

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