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Sinc-chebyshev Collocation Method for PDAEs

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Abstract The partial differential algebraic equations (PDAEs) occurs frequently in various applications in mathematical modeling, physical problems, multibody mechanics, spacecraft control and incompressible fluid dynamics. In recent years, PDAEs have received much attention, nevertheless the theory in this field is still young.

In this paper, we introduce some results in the research of PDAEs. What is more, we propose a sinc-chebyshev collocation method for the time dependent linear PDAEs.

Keywords PDAEs; Collocation method; Sinc-chebyshev collocation

1. Introduction

Considering the linear partial differential algebraic equations(PDAEs) of the form:

$$A\frac{\partial U(x,t)}{\partial t}+B\frac{\partial^2 U(x,t)}{\partial x^2}+C\frac{\partial U(x,t)}{\partial x}+DU(x,t)=f(x,t), \tag{1}$$

where $t\in[t_0,T], A,B,C,D\in\mathbb{R}^{n\times n}$ and $U,f:[t_0,T]\times(a,b)\to R^m$. The interest is in the case where the matrix A and B are singular. The singularity of A leads to the differential-algebraic aspect. The above differential equation is required to satisfy the following boundary and initial conditions:

$$EU(x,t)+F\frac{\partial U(x,t)}{\partial v}=g(x,t), x\in\Gamma, t\in[t_0,T], \tag{2}$$

$$U(x,t_0)=U_0(x), \tag{3}$$

where E and F are known constant matrices and $g(x,t):[t_0,T]\times(a,b)\to R^m$ and $U_0(x)$are known functions, $\frac{\partial}{\partial v}$ is the outward normal derivative such that $\frac{\partial}{\partial v}=-\frac{\partial}{\partial x}$ at the left boundary $x=a$ and $\frac{\partial}{\partial v}=\frac{\partial}{\partial x}$ at the right boundary $x=b$. When $F=0$ and $E\neq 0$, we call it Dirichlet type boundary condition, when $E=0$ and $F\neq 0$, we call it Neumann type boundary condition and when $E\neq 0$ and $F\neq 0$, we call it mixed type or Robin type boundary condition. Many important mathematical models can be expressed in terms of PDAEs. Such models arise in many areas of mathematics, engineering, physical sciences and population growth. In recent years, PDAEs have received much attention, nevertheless the theory in this field is still young. In paper [13], Debrabant and Strehmel applied Runge-Kutta methods to linear PDAEs and they proved that under certain conditions the temporal convergence order of the fully discrete scheme depends on the time index of the PDAEs. In paper [9], Bao and song proposed the meshless approach for the numerical solution of time dependent PDAEs in terms of finite differences scheme generated from radial basis functions (RBF-FD. In paper [7], they proposed two meshless collocation approaches for solving time dependent PDAEs in terms of the multi-quadric quasi-interpolation schemes.

In [1], huang and zhang invited the spectral collocation methods for the DAEs. In [14] the collocation method for solving linear DAE boundary value problems was developed. In [15], the authors constructed Sinc-collocation method for Fredholm integral equations.

In this paper, a sinc- chebyshev collocation scheme is explored for time dependent linear PDAEs. This approach is based on the collocation technique where the shifted chebyshev polynomials in time and the sinc function in space are utilized,respectively. The systems of PDAEs is reduced to a system of linear algebraic equations.To show the validity and efficiency of the present method, some examples are presented.

2 Sinc-chebyshev collocation method for PDAEs

In this section, we introduce some basic notations, definitions of the sinc function and the chebyshev polynomial and derive some known results and useful formulas for the new method.

2.1. Sinc function

The sinc collocation method,which was developed by F.Stenger more than twenty years ago [11], is based on the Whittaker-Shannon-Kotel'nikol sampling theorem for entire function as bases,has many advantages over classical methods that use polynomials as bases. The sinc function used is

$$sinc(t) = \begin{cases} \frac{sin(\pi t)}{\pi t}, t \neq 0 \\ 1, t = 0 \end{cases}$$

and the set of basis function adopted are:

$$S(j,h)(t) = sinc(\frac{t-jh}{h}) = \begin{cases} \frac{sin(\frac{\pi(t-jh)}{h})}{\frac{\pi(t-jh)}{h}}, t \neq jh \\ 1, t = jh, j \in \mathbb{Z} \end{cases}$$

where h is the step-size and $\mathbb{Z}$ denotes the set of all integers[11]. The sinc functions form an interpolator set of functions, i.e.

$$S(j,h)(kh) = \delta_{jk} = \begin{cases} 1, j = k \\ 0, j \neq k \end{cases}$$

where δ_{jk} is Kronecker delta function.If a function $f(t)$ is defined on the real line, then for any $h > 0$ the series

$$C(f,h)(t) = \sum_{j=-\infty}^{\infty} f(jh)sinc(\frac{t-jh}{h})$$

is called the Whittaker cardinal expansion of $f(t)$ whenever the series converges.

To construct an approximation on the interval $[a,b]$, we choose the conformal map

$$\phi(x) = ln(\frac{x-a}{b-x})$$

which maps the finite interval $[a,b]$ onto $\mathbb{R}$. The basis function on $[a,b]$ are taken to be the composite translated sinc functions:

$$S_\phi(j,h)(x) = S(j,h)(\phi(x)) = sinc(\frac{\phi(x)-jh}{h})$$

and these functions exhibit Kronecker delta behavior on the grid point $x_k \in (a,b)$ defined by:

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, j = 0, \pm1, \pm2, ...$$

For further explanation of the procedure,we consider the following definition and theorem in [11] and [12]:

Definition 2.1 *Let $B(D_E)$ be the class of function f which are analytic in D_E and satisfy*

$$\int_{\phi^{-1}(x+L)} |f(z)dz| \to 0, x \to \pm\infty,$$

where $L = iv: |v| < d \leq \frac{\pi}{2}$, and

$$\int_{\partial D_E} |f(z)dz| < \infty$$

on the boundary of D_E.

Lemma 2.2 *If $\phi' f \in B(D_E)$, then for all $x \in [a,b]$*

$$|f(x) - \sum_{k=-\infty}^{\infty} f(x_k)S_\phi(k,h)(x)| \leq \frac{2N(\phi' f)}{\pi d} e^{-\frac{\pi d}{h}}$$

Further, one assumes that there are positive constant C and β so that $|f(x)| \leq Cexp(-\beta|\phi(x)|)$. And if one selects $h = \sqrt{\frac{\pi d}{\beta N}} \leq \frac{2\pi d}{ln 2}$, then

$$\left|\frac{d^m f(x)}{dx^m} - \sum_{k=-N}^{N} f(x_k)\frac{d^m}{dx^m} S_\phi(k,h)(x)\right| \le KN^{(m+1)/2} exp(-\sqrt{\pi d\beta N})$$

for all $m = 0,1,\dots n$.

We also require derivatives of composite sinc functions evaluated at the nodes:

$$\delta_{jk}^{(n)} = \frac{d^n}{d\phi^n}[S_\phi(j,h)(x)]|_{x=x_k}.$$

In particular

$$\delta_{jk}^{(0)} = [S_\phi(j,h)(x)]|_{x=x_k} = \begin{cases} 0, j \ne k \\ 1, j = k. \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S_\phi(j,h)(x)]|_{x=x_k} = \begin{cases} \frac{(-1)^{k-j}}{(k-j)}, j \ne k \\ 0, j = k. \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S_\phi(j,h)(x)]|_{x=x_k} = \begin{cases} \frac{-2(-1)^{k-j}}{(k-j)^2}, j \ne k \\ \frac{-\pi^2}{3}, j = k. \end{cases}$$

2.2. General orthogonal polynomials

The general orthogonal polynomials plays a center role throughout the paper. It is well-known that Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is a class of orthogonal polynomial. The associated inner product can be defined as

$$< u(t), v(t) >= \int_{-1}^{1} \omega(t)u(t)v(t)dt$$

with a weight function $\omega(x) = (1-x)^\alpha(1+x)^\beta, \alpha, \beta > -1$:

$$P_k^{(\alpha,\beta)}(x) = \frac{1}{2^k}\sum_{l=0}^{k}\binom{k+\alpha}{k-l}(k+\beta l)(x-1)^l(x+1)^{k-l},$$

Especially, Legendre polynomial is a class of Jacobi polynomial with $\alpha = \beta = 0$ and Chebyshev polynomial of the first kind is another class with $\alpha = \beta = -\frac{1}{2}$.

2.3. The sinc-chebyshev collocation method

This section is devoted to the sinc-chebyshev collocation method for the time dependent PDAEs and its convergence analysis. First of all, we approximate $U(x,t)$ by the sinc function in space and the chebyshev polynomials in time:

$$U_{M,N}(x,t) = \begin{pmatrix} U_{M,N}^1 \\ U_{M,N}^2 \\ \vdots \\ U_{M,N}^n \end{pmatrix} = \begin{pmatrix} \sum_{i=-M}^{M}\sum_{j=0}^{N} c_{i,j}^1 S_\phi(i,h)(x)T_{\tau,j}(t) \\ \sum_{i=-M}^{M}\sum_{j=0}^{N} c_{i,j}^2 S_\phi(i,h)(x)T_{\tau,j}(t) \\ \vdots \\ \sum_{i=-M}^{M}\sum_{j=0}^{N} c_{i,j}^n S_\phi(i,h)(x)T_{\tau,j}(t) \end{pmatrix} \quad (4)$$

In order to discretizing the PDAEs, the lemma is given as follows:

Lemma 2.3 *Let x_k be the spatial collocation points, then*

$$\frac{\partial U_{M,N}}{\partial t}(x_k,t) = \begin{pmatrix} \sum_{j=0}^{N} c_{k,j}^1 T'_{\tau,j}(t) \\ \sum_{j=0}^{N} c_{k,j}^2 T'_{\tau,j}(t) \\ \vdots \\ \sum_{j=0}^{N} c_{k,j}^n T'_{\tau,j}(t) \end{pmatrix} \quad (5)$$

$$\frac{\partial U_{M,N}}{\partial x}(x_k,t)=\begin{pmatrix}\sum_{i=-M}^{M}\sum_{j=0}^{N}c_{i,j}^{1}\phi'(x_k)\delta_{i,k}^{(1)}T_{\tau,j}(t)\\ \sum_{i=-M}^{M}\sum_{j=0}^{N}c_{i,j}^{2}\phi'(x_k)\delta_{i,k}^{(1)}T_{\tau,j}(t)\\ \vdots\\ \sum_{i=-M}^{M}\sum_{j=0}^{N}c_{i,j}^{n}\phi'(x_k)\delta_{i,k}^{(1)}T_{\tau,j}(t)\end{pmatrix} \tag{6}$$

$$\frac{\partial^2 U_{M,N}}{\partial x^2}(x_k,t)=\begin{pmatrix}\sum_{i=-M}^{M}\sum_{j=0}^{N}c_{i,j}^{1}(\phi''(x_k)\delta_{i,k}^{(1)})+\phi'^2(x_k)\delta_{i,k}^{(2)})T_{\tau,j}(t)\\ \sum_{i=-M}^{M}\sum_{j=0}^{N}c_{i,j}^{2}(\phi''(x_k)\delta_{i,k}^{(1)})+\phi'^2(x_k)\delta_{i,k}^{(2)})T_{\tau,j}(t)\\ \vdots\\ \sum_{i=-M}^{M}\sum_{j=0}^{N}c_{i,j}^{n}(\phi''(x_k)\delta_{i,k}^{(1)})+\phi'^2(x_k)\delta_{i,k}^{(2)})T_{\tau,j}(t)\end{pmatrix} \tag{7}$$

Let coefficients

$$\Lambda^p=\begin{pmatrix}c_{-M,0}^{p} & c_{-M,1}^{p} & \cdots & c_{-M,N}^{p}\\ c_{-M+1,0}^{p} & c_{-M+1,1}^{p} & \cdots & c_{-M+1,N}^{p}\\ \vdots & \vdots & \cdots & \vdots\\ c_{M,0}^{p} & c_{M,1}^{p} & \cdots & c_{M,N}^{p}\end{pmatrix}, p=1,2\cdots n$$

In order to determine the coefficients, we use the roots $t_l, l=1,2,\ldots N$ of the shifted Chebyshev polynomials $T_{\tau,N-1}$.

The PDAES

$$A\frac{\partial U(x,t)}{\partial t}+B\frac{\partial^2 U(x,t)}{\partial x^2}+C\frac{\partial U(x,t)}{\partial x}+DU(x,t)=f(x,t),$$

can be written as:

$$A\Lambda_k T'_l+B\begin{pmatrix}P_k\Lambda^1 T_l\\ P_k\Lambda^2 T_l\\ \vdots\\ P_k\Lambda^N T_l\end{pmatrix}+C\begin{pmatrix}Q_k\Lambda^1 T_l\\ Q_k\Lambda^2 T_l\\ \vdots\\ Q_k\Lambda^N T_l\end{pmatrix}+D\Lambda_k T_l=f(x_k,t_l) \tag{8}$$

where

$$\Lambda_k=\begin{pmatrix}c_{k,0}^{1} & c_{k,1}^{1} & \cdots & c_{k,N}^{1}\\ c_{k,0}^{2} & c_{k,1}^{2} & \cdots & c_{k,N}^{2}\\ \vdots & \vdots & \cdots & \vdots\\ c_{k,0}^{n} & c_{k,1}^{n} & \cdots & c_{k,N}^{n}\end{pmatrix}, T_l=\begin{pmatrix}T_{\tau,0}(t_l)\\ T_{\tau,1}(t_l)\\ \vdots\\ T_{\tau,N}(t_l)\end{pmatrix},$$

$$P_k=(p_{-M,k},p_{-M+1,k},\cdots,p_{M,k}), where p_{i,k}=\phi'(x_k)\delta_{ik}^{(1)},$$

$$Q_k=(q_{-M,k},q_{-M+1,k},\cdots,q_{M,k}), where q_{i,k}=\phi''(x_k)\delta_{ik}^{(1)}+\phi'^2(x_k)\delta_{ik}^{(2)},$$

The boundary condition and the initial condition in $2M+1$ points x_k can be written as:

$$\begin{pmatrix}U^1(x_k,0)\\ U^2(x_k,0)\\ \vdots\\ U^n(x_k,0)\end{pmatrix}=\begin{pmatrix}\sum_{j=0}^{N}(-1)^j c_{k,j}^{1}\\ \sum_{j=0}^{N}(-1)^j c_{k,j}^{2}\\ \vdots\\ \sum_{j=0}^{N}(-1)^j c_{k,j}^{n}\end{pmatrix}=\begin{pmatrix}g^1(x_k)\\ g^2(x_k)\\ \vdots\\ g^n(x_k)\end{pmatrix} \tag{9}$$

$$\begin{pmatrix}\frac{\partial U^1}{\partial t}(x_k,0)\\ \frac{\partial U^2}{\partial t}(x_k,0)\\ \vdots\\ \frac{\partial U^n}{\partial t}(x_k,0)\end{pmatrix}=\begin{pmatrix}\sum_{j=0}^{N}(-1)^{j-1}\frac{2j^2}{\tau}c_{k,j}^{1}\\ \sum_{j=0}^{N}(-1)^{j-1}\frac{2j^2}{\tau}c_{k,j}^{2}\\ \vdots\\ \sum_{j=0}^{N}(-1)^{j-1}\frac{2j^2}{\tau}c_{k,j}^{n}\end{pmatrix}=\begin{pmatrix}h^1(x_k)\\ h^2(x_k)\\ \vdots\\ h^n(x_k)\end{pmatrix} \tag{10}$$

Applying vec operator to (3.5 3.7), we obtain

$$(T_l'^T\otimes A+T_l^T\otimes D)vec\Lambda_k+(T_l^T\otimes(B_1P_k+C_1Q_k),\cdots,T_l^T\otimes(B_nP_k+C_nQ_k))\begin{pmatrix}vec\Lambda^1\\ vec\Lambda^2\\ \vdots\\ vec\Lambda^n\end{pmatrix}=f(x_k,t_l) \tag{11}$$

$$(W_1^T \otimes I)vec\Lambda_k = \begin{pmatrix} g^1(x_k) \\ g^2(x_k) \\ \vdots \\ g^n(x_k) \end{pmatrix}, where W_1 = (1,-1,1,\dots,(-1)^N) \tag{12}$$

$$(W_2^T \otimes I)vec\Lambda_k = \begin{pmatrix} h^1(x_k) \\ h^2(x_k) \\ \vdots \\ h^n(x_k) \end{pmatrix}, where W_2 = (1,1,1,\dots,(1)^N) \tag{13}$$

Thus, from equations$(3.8) - (3.10)$ we know that Λ^k can be obtained by solving the linear equations:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \begin{pmatrix} vec\Lambda^1 \\ vec\Lambda^2 \\ \vdots \\ vec\Lambda^n \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

Here vec is the vec operator such that $vec(\Lambda^p)$ is the vector of columns of Λ^p stacked one under the other and $\otimes$ denotes the Kronecker product. Once $vec(\Lambda^p)$ is determined, we plug it back to equations(3.1) to obtain $U_{M,N}(x,t)$.

3. Conclusions and future work

By discretizing a class of initial problem of PDAEs with the sinc-collocation method, we have obtained a new numerical method for PDAEs and analyzed the structure of the method. In the process of getting the numerical solution by the above method, we need to solve the system of linear equations with special structures. Our future work will focus on extending the application scopes of these theorems.

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