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Stiffness Matrix Method for Nonlinear Analysis of Plane Frames

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Abstract In this paper, geometric nonlinear analysis of plane frames was performed by the stiffness matrix method using stability functions. At first, the argument of the stability functions was set as 0.01. The stiffness matrix of the frame has been assembled, as well as the nodal load vector of the frame. The boundary conditions (support restraint and windbracing restraint) were introduced for the reduction of this matrix and the nodal load vector. At this stage, the determinant of the reduced stiffness matrix and the reduced nodal displacement vector are calculated. The argument of the stability functions is incremented by 0.01 and the operations are repeated until the determinant of the reduced stiffness matrix changes sign. The argument of the iteration preceding the sign change of the determinant and corresponding to its positive value is taken and refined by a process described in the paper. The buckling loads of the frame members are determined at this stage. Windbracing and the increase of supports stiffness increase the value of the critical load (less sensitivity to the phenomena of elastic instability) and have been identified as factors of stability.

Keywords geometric nonlinear analysis, stiffness matrix method, stability functions, determinant, buckling load

1. Introduction

The geometric nonlinear analysis of the plane frames will be made by the stiffness matrix method using stability functions. Geometric nonlinear analysis of frames is a problem of elastic instability. Although the structure is in the elastic range, the effects of large displacements (large deformations) modify the geometry of the structure; which renders inapplicable the theory of linear elasticity. Examples of these phenomena of instability are buckling, local buckling and lateral buckling [1].

Stability analysis in solid mechanics began with Euler's solution of buckling of an elastic column (Euler, 1744). Most basic linear elastic problems of structural stability were solved by the end of the 19th century, although further solutions have been appearing as new structural types were being introduced. The twentieth century has witnessed a great expansion of the stability theory into nonlinear behavior, caused either by large deflections or by nonlinearity of the constitutive law of the material [2].

Several authors have studied stiffness matrix method and nonlinear analysis of structures. Some of these studies include: Sezer (1995) did some works on nonlinear analysis of reinforced concrete plates by finite element method [3]; Gummandi and Palazotto (1997) performed nonlinear analysis of beams and arches by finite element method [4]; Bhatt (1999) worked on the elastic stability of structural systems by stiffness matrix method [5]; Ozer (2006) used various methods for nonlinear analysis of structural systems [6]. Orumu (2013) has proposed mathematical formulae for estimating the critical loads of rigid sway frames with pinned or fixed base restraints [7].

The aim of the study is to find the critical loads of the frame members. The determinant of the reduced stiffness matrix of the frame, as well as the argument of the stability functions are used to determine these critical loads. A computer program is used to carry out the various operations.

2. Materials and Method

2.1. Geometry, loading and support conditions

This paper is devoted to calculating the critical load of rectangular plane frames made of straight members. The posts of the frame, which are vertical, rise continuously from the foundation to the top of the structure. Similarly, the beams (horizontal) are continuous from the left to the right of the structure. Thus, there is no missing bar in an interior panel. The posts of the first storey can be of different lengths. The bars are connected to the nodes, which are considered non-deformable, by rigid assemblies. The feet of the posts can be pinned or fixed in a rigid manner to the foundation.

The external forces considered in the present study are such that, before the loss of stability, the members undergo only axial compression (or traction). By way of example, Figure 1 shows such a system of forces. Here, the loads are only concentrated loads applied to the nodes, acting in the axis of the posts.

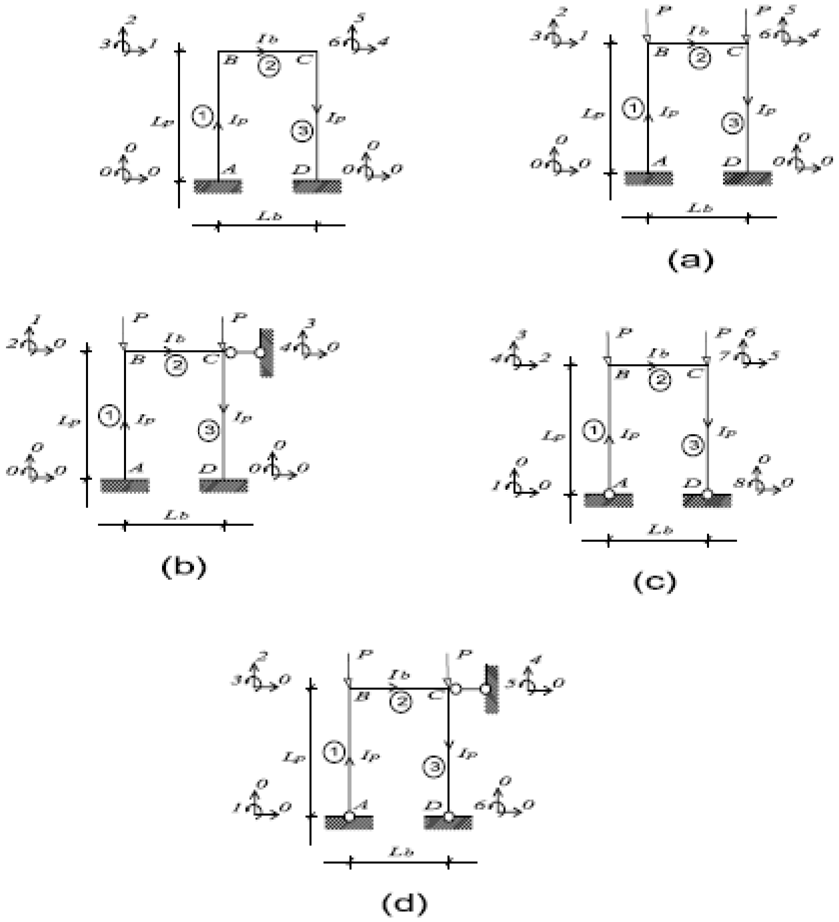

Figure 1: Categories of structures studied

A frame of a building is, in practice, subject to loads due to self-weight and overloads. The transformation of dead weights and overloads acting on the horizontal members into nodal forces is done by a simple process. Each beam is considered between two consecutive nodes as a beam on simple supports. The equivalent concentrated loads acting at the nodes are equal and of opposite sign to the reactions of this simple beam. When there are horizontal forces acting at the nodes, the axial forces in the members are found by a preliminary calculation.

2.2. Assumptions

This study is based on the following assumptions:

- The materials that make up the bars are supposed to be perfectly elastic.
- The nodes are supposed to be rigid.
- Forces are expected to maintain their original direction and initial point of application during buckling.

- In addition, the external forces are such that, before the loss of stability, the bars undergo only an axial force (compression or traction).
- Deformations (in the displaced state) are considered small.
- The case of buckling studied is that of bending buckling in the plane of the frame. It is assumed that the spatial buckling accompanied by twisting and local buckling of the walls are prevented.

In addition, the elastic length variations due to axial forces are neglected. All the nodes of a crossbar thus have the same transverse displacement.

2.3. Stability study

Nonlinear analysis of structures can be formulated using Figure 2 and Equation 1. The stiffness matrix ($[k_i]$ in local coordinate system) and that of rotation transformation, $[T_i]$, of a member << *i* >> are modified during the increase of the axial load. However, in this study, $[T_i]$ is assumed unaffected.

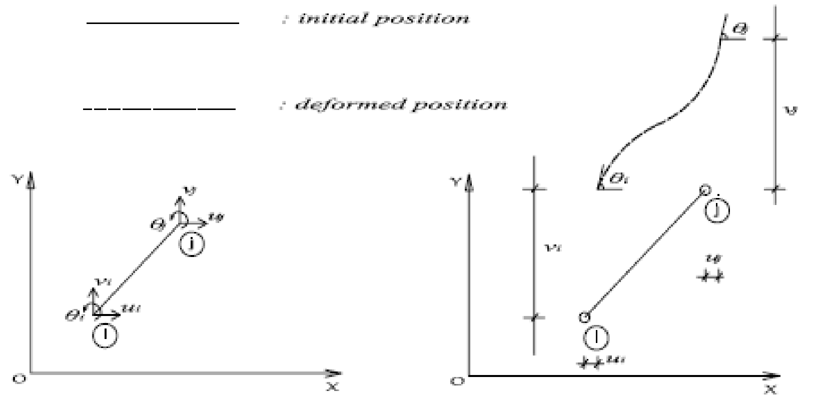

Figure 2: Displaced position and degrees of freedom (d.o.f.) of a member.

$$\{f_i\} = \left[k_i\right]\{d_i\} \tag{1}$$

Where:

$$\{f_i\} = \begin{bmatrix} P_i \\ V_i \\ M_i \\ P_j \\ V_j \\ M_j \end{bmatrix} \quad and \quad \{d_i\} = \begin{bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{bmatrix}$$

$\{f_i\}$ is the nodal load vector of the member; P_i, V_i and M_i respectively represent axial force, shear force and bending moment at node *i*.

$\{d_i\}$ is the vector of nodal deformations of the member; u_i, v_i and θ_i respectively represent axial displacement , lateral displacement and rotation at node *i*.

$$\left[k_i\right] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3}.\eta_2(u) & \frac{6EI}{L^2}.\varphi_4(u) & 0 & -\frac{12EI}{L^3}.\eta_2(u) & \frac{6EI}{L^2}.\varphi_4(u) \\ 0 & \frac{6EI}{L^2}.\varphi_4(u) & \frac{4EI}{L}.\varphi_2(u) & 0 & -\frac{6EI}{L^2}.\varphi_4(u) & \frac{2EI}{L}.\varphi_3(u) \\ \frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3}.\eta_2(u) & -\frac{6EI}{L^2}.\varphi_4(u) & 0 & \frac{12EI}{L^3}.\eta_2(u) & -\frac{6EI}{L^2}.\varphi_4(u) \\ 0 & \frac{6EI}{L^2}.\varphi_4(u) & \frac{2EI}{L}.\varphi_3(u) & 0 & -\frac{6EI}{L^2}.\varphi_4(u) & \frac{4EI}{L}.\varphi_2(u) \end{bmatrix}$$

The stability functions [8] of the members (equation 2) cannot be calculated because the axial forces acting on them are unknown. An iterative method is used to overcome this difficulty. One starts by calculating these functions using as load argument υ = 0.01.

The following steps should be implemented for the nonlinear analysis of a frame:

Step 1: Idealize the structure and establish global axes,

Step 2: Number the nodes (A, B, C, D, ...) and the degrees of freedom (*d.o.f.*) of the structure (0 for *inactive d.o.f.*, and 1, 2, 3, 4, 5, 6, 7, …, for the others),

Step 3: Number the members and assign an arrow to each member so that ends *i* and *j* are defined,

Step 4: Enter the geometric characteristics (area, inertia, length) of each member and its orientation angle, θ, as well as the mechanical properties of the materials (YOUNG's modulus, COULOMB's modulus),

$$
\begin{aligned}
\varphi_1(\upsilon) &= \frac{\upsilon^2 \tan\upsilon}{3(\tan\upsilon - \upsilon)} \\
\varphi_2(\upsilon) &= \frac{\upsilon(\tan\upsilon - \upsilon)}{8\tan\upsilon(\tan\frac{\upsilon}{2} - \frac{\upsilon}{2})} \\
\varphi_3(\upsilon) &= \frac{\upsilon(\upsilon - \sin\upsilon)}{4\sin\upsilon(\tan\frac{\upsilon}{2} - \frac{\upsilon}{2})} \\
\varphi_4(\upsilon) &= \varphi_1(\frac{\upsilon}{2}) \\
\eta_1(\upsilon) &= \frac{\upsilon^3}{3(\tan\upsilon - \upsilon)} \\
\eta_2(\upsilon) &= \eta_1(\frac{\upsilon}{2})
\end{aligned}
\tag{2}
$$

Step 5: Initialize the argument υ to 0.01 and the iterations counter << *Count.* >> to 1,

Step 6: Enter the expressions of the stability functions and form the stiffness matrix, $[k_i]$, and the nodal load vector, $\{f_i\}$, of each member in the local coordinate system; for uncompressed members,

$$\varphi(\upsilon) = \eta(\upsilon) = 1 \tag{3}$$

Step 7: Form the rotation transformation matrix, $[T_i]$, of each member and calculate its stiffness matrix, $[K_i]$, and its nodal load vector, $\{F_i\}$, in the global coordinate system,

$$\left[T_i\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\left[K_i\right] = \left[T_i\right]^T . \left[k_i\right] \left[T_i\right] \tag{5}$$

Step 8: Assemble the stiffness matrix, $[K]$, and the nodal load vector, $\{F\}$, of the structure,

Step 9: Form the reduced stiffness matrix, $[K_r]$, and the reduced nodal load vector, $\{F_r\}$, of the structure by ignoring the rows and columns corresponding to *inactive d.o.f.* (restraint of supports and restraint of windbracing) and of zero order number << 0 >>,

Step 10: Calculate the reduced nodal deformation vector, $\{D_r\}$, of the structure:

$$\{D_r\} = \left[K_r\right]^{-1} . \{F_r\} \tag{6}$$

Step 11: Calculate the determinant of the reduced stiffness matrix, $|K_r|$,

- If $|K_r|$ sign changes, go to **step 12**
- If not, increment υ and *Count.*, and return to **step 6**:

$$\upsilon = \upsilon + 0.01 \qquad and \qquad Count. = Count. + 1 \tag{7}$$

Step 12: Continue the calculations with the penultimate value of the argument υ, with a small increment (0.001) to improve the accuracy in determining the critical argument, υ_{cr}, and therefore, the critical load, P_{cr}; stop operations as soon as $|K_r|$ changes sign, and take υ for which $|K_r| > 0$, as critical argument, υ_{cr}.

2.4. Flowchart of the computer program

The algorithm required for the nonlinear analysis of plane frames (by stiffness matrix method) is represented in the form of a flowchart (Figure 3) showing the important tasks to be performed.

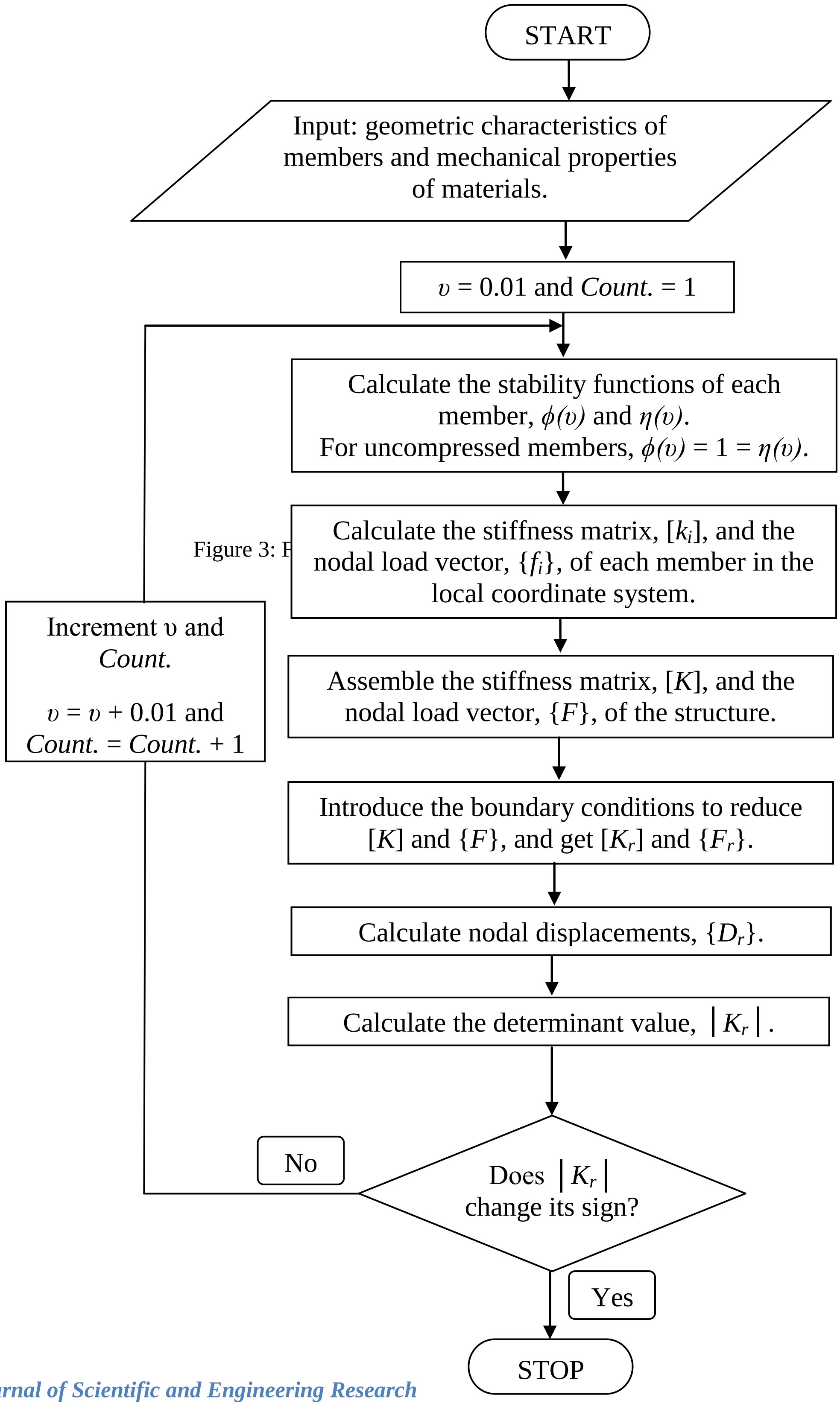

Figure 3: F

3. Numerical examples

3.1. Example 1

Let's start with a digital application for a basic construction. Let us study the stability of the simple symmetrical frame and symmetrically loaded shown in Figure 1a for the numerical values:

L_p = 12.00 m, L_b = 10.00 m, $I_p = I_b$ = 18260 cm^4, E = 21000 kN/cm^2, A = 331 cm^2, P = 1000 kN.

The following table presents the computation of the determinant $|K_r|$ and the nodal displacements $\{D_r\}$ of the frame, for increasing values of υ:

Table 1: Determinant and nodal displacements of the frame 1

No.	υ	Load *P* (kN)	$\lvert K_r \rvert$	Nodal displacements $\{D_r\}$ of the frame, d.o.f.	*u (mm)*	*v (mm)*	*θ (rad)*
1	0.01	2.66E-02	6.65824E+22	Node A	0	0	0
				Node B	1.34102E-17	-4.59718E-05	-5.98879E-22
				Node C	1.34068E-17	-4.59718E-05	-6.00533E-22
				Node D	0	0	0
2	0.02	1.07E-01	6.65792E+22	Node A	0	0	0
				Node B	5.36951E-17	-0.000183887	-2.39882E-21
				Node C	5.36951E-17	-0.000183887	-2.39882E-21
				Node D	0	0	0
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
282	2.773	2.05E+03	1.37024E+19	Node A	0	0	0
				Node B	0	-3.535015094	0
				Node C	0	-3.535015094	0
				Node D	0	0	0
283	2.774	2.05E+03	-2.20118E+19	Node A	0	0	0
				Node B	1.81899E-12	-3.53756515	-2.77556E-17
				Node C	4.54747E-13	-3.53756515	-2.77556E-17
				Node D	0	0	0

The reduced stiffness matrix of the frame, $[K_r]$, is a function of the load because of the geometrical non-linearity; this is used as a stability criterion:

$$\begin{aligned} |K_r| &> 0 \Rightarrow \textit{stable equilibrium} \\ |K_r| &< 0 \Rightarrow \textit{instable equilibrium} \\ |K_r| &= 0 \Rightarrow \textit{neutral equilibrium and } P_{cr} \end{aligned} \quad (8)$$

The following table presents the calculation of some characteristics of the frame:

Table 2: Critical parameters of the frame 1

Element	Critical value υ_{cr}	P_{cr} (kN)	λ_{cr}	ρ	μ
Post	2.773	2.05E+03	2.047657493	0.77911218	1.132921981

The value υ_{cr} is close to that given by the table of Bleich's book [9]; (υ_{cr} = 2.775).

We have:

$$
\begin{aligned}
P_{cr} &= \frac{u_{cr}^2.E.I}{L^2} \quad : \quad critical \quad load \\
l_{cr} &= \frac{u_{cr}^2.E.I}{P.L^2} = \frac{P_{cr}}{P} \quad : \quad critical \quad load \quad parameter \\
r &= \frac{P_{cr}}{P_E} \quad where \quad P_E = \frac{p^2.E.I}{L^2} \quad : \quad Euler's \quad load \\
m &= \frac{p}{u_{cr}} \quad : \quad effective \quad length \quad coefficient
\end{aligned}
\tag{9}
$$

3.2. Example 2

Now let us consider the same construction as before, but for which the transverse displacement of the nodes is prevented (Figure 1b).

The following table presents the computation of the determinant $|K_r|$ and the nodal displacements $\{D_r\}$ of the frame, for increasing values of υ:

Table 3: Determinant and nodal displacements of the frame 2

No.	υ	Load P (kN)	$\lvert K_r \rvert$	Nodal displacements $\{D_r\}$ of the frame			
					d.o.f.		
					u (mm)	*v (mm)*	*θ (rad)*
1	0.01	2.66E-02	2.45727E+20	Node A	0	0	0
				Node B	0	-4.59718E-05	8.27181E-25
				Node C	0	-4.59718E-05	-4.1359E-25
				Node D	0	0	0
2	0.02	1.07E-01	2.45724E+20	Node A	0	0	0
				Node B	0	-0.000183887	1.65436E-24
				Node C	0	-0.000183887	-1.65436E-24
				Node D	0	0	0
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
514	5.093	6.91E+03	2.50735E+15	Node A	0	0	0
				Node B	0	-11.92446452	0
				Node C	0	-11.92446452	0
				Node D	0	0	0
515	5.094	6.91E+03	-8.98958E+16	Node A	0	0	0
				Node B	0	-11.92914767	8.67362E-19
				Node C	0	-11.92914767	-4.33681E-19
				Node D	0	0	0

The following table presents the calculation of some characteristics of the frame:

Table 4: Critical parameters of the frame 2

Element	Critical value υ_{cr}	P_{cr} (kN)	λ_{cr}	ρ	μ
Post	5.093	6.91E+03	6.907246073	2.628134619	0.61684521

The value υ_{cr} is close to that given by the table of Bleich's book [9]; (υ_{cr} = 5.095).

3.3. Example 3

Now let us consider the same construction as that of Example 1, but for which the supports are pinned (Figure 1c).

The following table presents the computation of the determinant $|K_r|$ and the nodal displacements $\{D_r\}$ of the frame, for increasing values of υ:

Table 5: Determinant and nodal displacements of the frame 3

No.	υ	Load *P* (kN)	$\|K_r\|$	Nodal displacements $\{D_r\}$ of the frame d.o.f.	*u (mm)*	*v (mm)*	*θ (rad)*
1	0.01	2.66E-02	2.01414E+36	Node A	0	0	-3.3087E-24
				Node B	3.3881E-20	-4.5972E-05	-1.6544E-24
				Node C	6.0986E-20	-4.5972E-05	-2.4815E-24
				Node D	0	0	-6.6174E-24
2	0.02	1.07E-01	2.01375E+36	Node A	0	0	-2.9779E-23
				Node B	2.4395E-19	-1.8389E-04	-3.3087E-24
				Node C	1.8974E-19	-1.8389E-04	-9.9262E-24
				Node D	0	0	-2.6470E-23
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
140	1.38	5.07E+02	2.28503E+33	Node A	0	0	-1.0608E-13
				Node B	9.0540E-10	-8.7549E-01	-2.0027E-14
				Node C	9.0540E-10	-8.7549E-01	-2.0022E-14
				Node D	0	0	-1.0610E-13
141	1.381	5.08E+02	-3.21836E+30	Node A	0	0	7.5403E-11
				Node B	-6.4349E-07	-8.7676E-01	1.4232E-11
				Node C	-6.4349E-07	-8.7676E-01	1.4227E-11
				Node D	0	0	7.5417E-11

The following table presents the calculation of some characteristics of the frame:

Table 6: Critical parameters of the frame 3

Element	Critical value υ_{cr}	P_{cr} (kN)	λ_{cr}	ρ	μ
Post	1.38	5.07E+02	0.50712585	0.192956062	2.276516416

3.4. Example 4

Now let us consider the same construction as before, but for which the transverse displacement of the nodes is prevented (Figure 1d).

The following table presents the computation of the determinant $|K_r|$ and the nodal displacements $\{D_r\}$ of the frame, for increasing values of υ:

Table 7: Determinant and nodal displacements of the frame 4

No.	υ	Load *P* (kN)	$\|K_r\|$	Nodal displacements $\{D_r\}$ of the frame d.o.f.	*u (mm)*	*v (mm)*	*θ (rad)*
1	0.01	2.66E-02	3.0846E+34	Node A	0	0	-1.2408E-24
				Node B	0	-4.5972E-05	1.6544E-24
				Node C	0	-4.5972E-05	0.0000E+00
				Node D	0	0	-1.0340E-24

2	0.02	1.07E-01	3.08449E+34	Node A	0	0	0.0000E+00
				Node B	0	-1.8389E-04	-1.6544E-24
				Node C	0	-1.8389E-04	-3.3087E-24
				Node D	0	0	-4.1359E-24
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
.	.	.	.	Node A	.	.	.
				Node B	.	.	.
				Node C	.	.	.
				Node D	.	.	.
375	3.649	3.55E+03	3.569E+30	Node A	0	0	7.4810E-17
				Node B	0	-6.1212E+00	-4.8681E-17
				Node C	0	-6.1212E+00	4.8572E-17
				Node D	0	0	-7.4593E-17
376	3.650	3.55E+03	-2.88946E+29	Node A	0	0	-9.1897E-16
				Node B	0	-6.1246E+00	6.0043E-16
				Node C	0	-6.1246E+00	-6.0043E-16
				Node D	0	0	9.1919E-16

The following table presents the calculation of some characteristics of the frame:

Table 8: Critical parameters of the frame 4

Element	**Critical value υ_{cr}**	**P_{cr} (kN)**	**λ_{cr}**	**ρ**	**μ**
Post	3.649	3.55E+03	3.545727066	1.349111926	0.860946192

4. Analysis of the results

The following figure shows the variation of the determinant of reduced stiffness matrix of the structure according to the upsilon argument for the four structures previously studied.

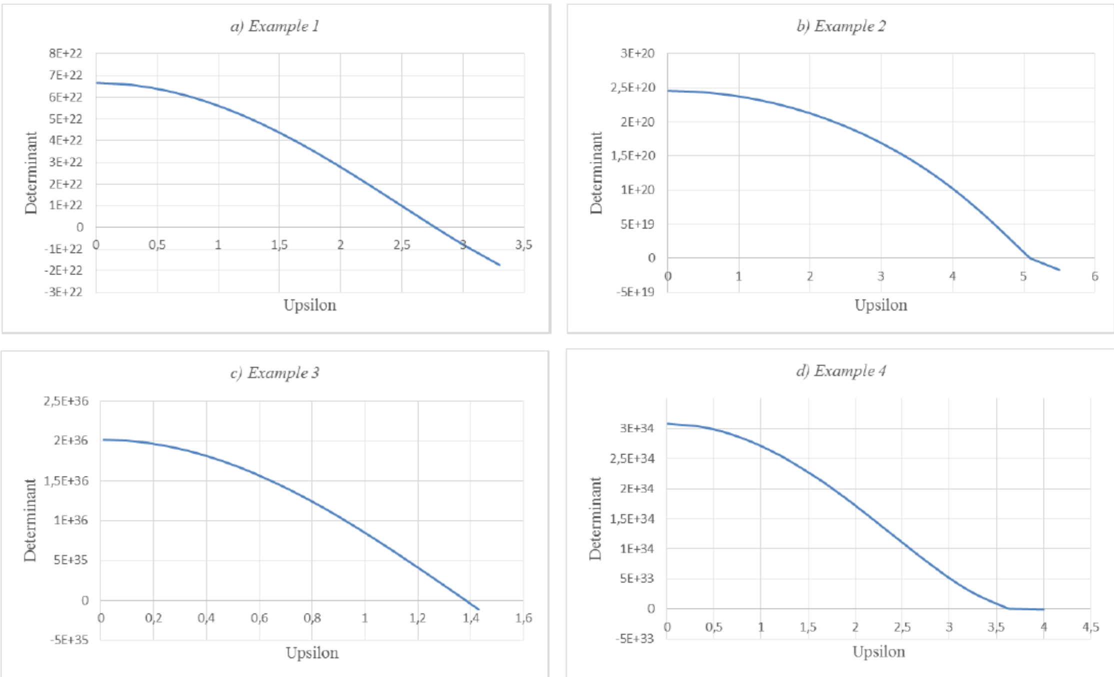

Figure 4: Determinant-upsilon curves

The following table summarizes the critical load values obtained for the four previous examples:

Table 9: Summary of critical load values

	Critical loads P_{cr} (kN)		Ratio
	Posts support conditions		$(P^{(f)}_{cr}/ P^{(p)}_{cr})$
	Fixed$^{(f)}$	Pinned$^{(p)}$	
Unbraced structure^1	2.05E+03	5.07E+02	**4.0434**
Braced structure^2	6.91E+03	3.55E+03	**1.9465**
Ratio (P^2_{cr}/ P^1_{cr})	**3.3707**	**7.0020**	

The analysis of the table reveals that windbracing or increase of supports stiffness increases the value of the critical load (less sensitivity to the phenomena of elastic instability).

The results obtained by Chajes [10] and Timoshenko [11] strengthen our position.

5. Conclusion

In this paper, we presented a simple and fast method for the nonlinear analysis of plane frames, using stability functions, and updating the stiffness matrix of the structure at each iteration. The reduced stiffness matrix determinant, $|K_r|$, and the stability function argument, υ, were used to check the singularity condition of the reduced stiffness matrix, $[K_r]$, and the iterations were stopped when $|K_r|$ changed sign. At this stage, the critical load and some parameters related to it have been determined.

Through the results obtained for the cases studied, the windbracing and the increase of the stiffness of the supports were identified as stability factors.

An important event observed during the analysis is that starting from the first iteration, i.e. "υ = 0.01", the determinant $|K_r|$ is positive until υ_{cr} (Figure 4); which reflects the stability of the structure for loads $P < P_{cr}$.

The analysis focused on four frames; the obtained results are similar to those present in the literature.

Nomenclature

A	cross-sectional area of a member (cm^2)
d.o.f.	degree of freedom
$\{d\}$	vector of nodal deformations
$\{D\}$	vector of nodal deformations of the structure
E	YOUNG's modulus of elasticity (kN/cm^2)
$\{F\}$	structural nodal load vector
$\{f\}$ and $\{F\}$	nodal load vectors
G	COULOMB's modulus of elasticity (kN/cm^2)
I	moment of inertia of a section (cm^4)
$[K]$	structural stiffness matrix
L	length of a member (cm)
M	bending moment
P	axial force
$[T]$	rotation transformation matrix
u	axial displacement
v	lateral displacement
V	shear force

Greek symbols

θ	rotation (positive counter-clockwise)
μ	effective length coefficient
υ	argument of stability functions
$\varphi(\upsilon)$ and $\eta(\upsilon)$	stability functions

Subscripts and superscripts

b	beam
cr	critical

i	member index, node index
f	fixed
p	pinned, post
r	reduced
T	transpose of a matrix

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