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***Journal of Scientific and Engineering Research*, 2018, 5(9):211-214**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Characterizing the exponential distribution by *m*-Spacings

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Abstract This paper is concerned with characterizing the exponential distribution by a property of the exponential m-spacings. We prove that an m-spacing of the order statistics corresponding to a random sample drawn from an exponential distribution is equal in distribution to a finite sum of independent heterogeneous exponentially distributed random variables. We use this characterization to prove some useful related identities.

Keywords exponential distribution, *m*-Spacings

1. Introduction

We say that a non-negative random variable X has an exponential distribution with parameter $\lambda > 0$, and we write $X \sim \text{Exp}(\lambda)$, if the probability density function (pdf) $f_X(x)$ of X is given by

$$f_X(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x), \tag{1.1}$$

where $I_A(x) = 1 \text{if} x \in A$ and $I_A(x) = 0 \text{if} x \notin A$.

If $X \sim \text{Exp}(\lambda)$, then its cumulative distribution function (cdf) $F_X(x)$ is given by

$$F_X(x) = 1 - e^{-\lambda x} I_{(0,\infty)}(x). \tag{1.2}$$

The exponential distribution is used to model various random times such as waiting time, failure time, duration time, inter-arrival time, etc. It has a lot of applications in may areas like reliability engineering, queueing theory, actuarial sciences, biology, and communication systems. Many authors studied the characterization of the exponential distribution by many different ways including the properties of order statistics, spacings of order statistics, moments of order statistics, the loss of memory property, and conditional expectations.

For a thorough review on the exponential distribution and its characterization, one may read [1] or [5].

For a fixed positive integer n, let $X_1, X_2, \ldots, X_n$ be independent random variables with common absolutely continuous cdf F_X. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics, where $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$. For all non-negative integers i, m, and n satisfying $0 \le i < m + i \le n$, define the sequence of m-spacings by $\mathcal{D}_{i,n}^{(m)} := X_{m+i:n} - X_{i:n}$. That is, the ith m-spacing, $\mathcal{D}_{i,n}^{(m)}$, is the gap between the $(m+i)$th and the ith order statistics. These spacings are also called m-step spacings or spacings of order m (see, for example,[6]). When $m = 1$, we will denote the 1-step spacings by $\mathcal{D}_{i,n}$ and simply call them *spacings*. By convention, we set $X_{0:n} = 0$ and, consequently, $\mathcal{D}_{0,n}^{(m)} = X_{m:n}$, for $1 \le m \le n$.

In the present work, we are interested in characterizing the exponential distribution by a property of the m-spacings. We use new techniques and approach to prove that For each set of non-negative integers i, m, and n satisfying $0 \le i < m + i \le n$, the ith exponential m-spacing $\mathcal{D}_{i,n}^{(m)}$ is distributed as the sum of independent heterogeneous exponentially distributed random variables. In other words, we prove that

$$\mathcal{D}_{i,n}^{(m)} \stackrel{d}{=} \sum_{j=1}^{m} X_j, \tag{1.3}$$

where $X_j, j = 1,2, \ldots, m$, are independent and $X_j \sim \text{Exp}((n - i - m + j)\lambda)$.

This representation enables us to provide straightforward proofs to many known and unknown characterizations of the exponential distribution.

2. The pdf of $\mathcal{D}_{i,n}^{(m)}$

The pdf of an m-spacing associated with the order statistics of a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ is give in [8]. We obtain the pdf of $\mathcal{D}_{i,n}^{(m)}$ of the exponential random variable of parameter $\lambda > 0$ by letting $\alpha = 1$ and $\beta = 1/\lambda$. In this case, the pdf of $\mathcal{D}_{i,n}^{(m)}$ is given as

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \sum_{k=0}^{m-1} \sum_{j=0}^{i-1} \binom{m-1}{k} \frac{(-1)^{i-j+m-k} n! \lambda e^{-(n-i-k)\lambda r}}{(m-1)! j! (i-1-j)! (n-i-m)! (n-j)}$$

$$= \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^{m-1-k} n! \lambda e^{-(n-i-k)\lambda r}}{(m-1)!(n-i-m)!} \sum_{j=0}^{i-1} \frac{(-1)^{i-j-1}}{j!(i-1-j)!(n-j)}, \tag{2.1}$$

$r > 0$.

Remark 2.1 *Since the spacings $\mathcal{D}_{i,n}^{(1)}$ are independent and $\mathcal{D}_{i,n}^{(m)} = \mathcal{D}_{i+1,n}^{(1)} + \cdots + \mathcal{D}_{i+m,n}^{(1)}$, the pdf of $\mathcal{D}_{i,n}^{(m)}$ can also be obtained by the convolution of the exponential random variables $\mathcal{D}_{i+1,n}^{(1)}, \ldots, \mathcal{D}_{i+m,n}^{(1)}$.*

Remark 2.2 *By the decomposition of $\prod_{j=1}^{i} \frac{1}{n+1-j}$ as a sum of partial fractions, for all positive integers i and n with $n \geq i \geq 1$, we obtain the identity*

$$\prod_{j=1}^{i} \frac{1}{n+1-j} = \sum_{j=0}^{i-1} \frac{(-1)^{i-1-j}}{j!(i-1-j)!(n-j)}. \tag{2.2}$$

Using mathematical induction on i, it is easy to show that

$$\prod_{j=1}^{i} \frac{1}{n+1-j} = \frac{(n-i)!}{n!}. \tag{2.3}$$

Using (2.3) in connection with (2.2), it follows that

$$\sum_{j=0}^{i-1} \frac{(-1)^{i-1-j}}{j!(i-1-j)!(n-j)} = \frac{(n-i)!}{n!}. \tag{2.4}$$

Remark 2.3 *Using (7), the pdf of $\mathcal{D}_{i,n}^{(m)}$ simplifies to*

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \sum_{k=0}^{m-1} m(-1)^{m-1-k} \binom{m-1}{k} \binom{n-i}{m} \lambda e^{-(n-i-k)\lambda r}. \tag{2.5}$$

3. m-Spacings as linear combinations of exponentials

Now, we are going to use a theorem from [4] that calculates the pdf of the sum $S_m := \sum_{i=1}^{m} X_i$ of the heterogeneous exponentially distributed random variables $X_1, \ldots, X_m$, where $X_i \sim \mathrm{Exp}(\lambda_i)$ and the parameters λ_i's are all distinct and strictly positive. The pdf of S_m is given by

$$f_{S_m}(t) = \sum_{i=1}^{m} \frac{\lambda_1 \cdots \lambda_m}{\prod_{\substack{j=1 \\ j \neq i}}^{m} (\lambda_j - \lambda_i)} \exp(-t\lambda_i), t > 0. \tag{3.1}$$

Let $\lambda_j = (n-i-m+j)\lambda$ for $j = 1,2,\cdots,m$. Then

$$\prod_{j=1}^{m} \lambda_j = \frac{(n-i)!\lambda^m}{(n-i-m)!}. \tag{3.2}$$

$$\prod_{\substack{j=1 \\ j \neq k}}^{m} (\lambda_j - \lambda_k) = (-1)^{m-1-k} \lambda^m k!(m-1-k)! = \frac{k!(m-1-k)!\lambda^m}{(-1)^{m-1-k}}. \tag{3.3}$$

Theorem 3.1 *Let $\mathcal{D}_{i,n}^{(m)}$ be the ith m-spacing corresponding to a random sample of size n from an exponential distribution with parameter $\lambda > 0$, where i, m, and n are non-negative integers satisfying $0 \leq i < m + i \leq n$. Then $\mathcal{D}_{i,n}^{(m)} \overset{d}{=} \sum_{j=1}^{m} X_j$, where $X_j \sim Exp((n-i-m+j)\lambda)$ and the X_j's are independent.*

Proof. Let $r > 0$. Then by (2.5)

$$f_{\mathcal{D}_{i,n}^{(m)}}(r) = \sum_{k=0}^{m-1} m(-1)^{m-1-k} \frac{(m-1)!}{k!(m-1-k)!} \frac{(n-i)!\lambda e^{-(n-i-k)\lambda r}}{m!(n-i-m)!}$$

$$= \sum_{k=0}^{m-1} \frac{(-1)^{m-1-k}}{k!(m-1-k)!\lambda^m} \frac{(n-i)!\lambda^m}{(n-i-m)!} \lambda e^{-(n-i-k)\lambda r}$$

$$= \sum_{k=0}^{m-1} \frac{\lambda_1 \lambda_2 \cdots \lambda_m}{\prod_{\substack{j=1 \\ j \neq k}}^{m} (\lambda_j - \lambda_k)} \lambda e^{-(n-i-k)\lambda r} \quad \text{(by (3.2) and (3.3))}$$

$$= f_{S_m}(r) \quad \text{(by (3.1))}.$$

4. Characterization by m-Spacings

Here, we will use a result by Desu (see [7]) that characterizes the exponential distribution by the property

$$kX_{1:k} \stackrel{d}{=} X, \tag{4.1}$$

for each positive integer $k \geq 1$, where $X \sim \text{Exp}(\lambda)$.

Theorem 4.1 *Let X be a non-negative random variable with an absolutely continuous cdf F_X. Let $\mathcal{D}_{i,n}^{(m)}$ be the sequence of m-spacings associated with the order statistics of a random sample of size n from F_X, where i, m, and n are non-negative integers satisfying $0 \leq i < i+m \leq n$. Then X is exponentially distributed with parameter $\lambda > 0$ if and only if its ith m-spacing $\mathcal{D}_{i,n}^{(m)} \stackrel{d}{=} \sum_{j=1}^{m} X_j$, where $X_j \sim Exp((n-i-m+j)\lambda)$ and the X_j's are independent.*

Proof. To prove the sufficient part, suppose that $X \sim \text{Exp}(\lambda)$. Then by Theorem 3.1,

$$\mathcal{D}_{i,n}^{(m)} := X_{m+i:n} - X_{i:n} \stackrel{d}{=} \sum_{j=1}^{m} X_j, \text{where} X_j \sim \text{Exp}((n-i-m+j)\lambda). \tag{4.2}$$

For the proof of the necessary part, suppose that

$$\mathcal{D}_{i,n}^{(m)} \stackrel{d}{=} \sum_{j=1}^{m} X_j \tag{4.3}$$

for all non-negative integers i, m, and n, satisfying $0 \leq i < i+m \leq n$.

Then $X_{m+i:n} - X_{i:n} \stackrel{d}{=} \sum_{j=1}^{m} X_j$. By letting $i = 0$ and $m = 1$, we see that $\mathcal{D}_{0,n}^{(1)} := X_{1:n} \stackrel{d}{=} X_1$, where $X_1 \sim \text{Exp}(n\lambda)$, which is equivalent to saying that $nX_{1:n} \sim \text{Exp}(\lambda)$. By (4.1), it follows that X has an exponential distribution with parameter λ.

Corollary 4.1 *When $m = 1$, the spacing $X_{i+1:n} - X_{i:n} \stackrel{d}{=} X_{n-i} \stackrel{d}{=} (n-i)X_1$, where $X_1 \sim Exp(\lambda)$.*

The following Corollary directly follows from (4.2).

Corollary 4.2 *The sequence $\mathcal{D}_{i,n}^{(m)}$ of m-spacings of order statistics corresponding to a random sample of size n from an exponential distribution with parameter $\lambda > 0$ satisfis*

$$\mathcal{D}_{i,n}^{(m)} \stackrel{d}{=} \mathcal{D}_{i+r,n+r}^{(m)}, \text{for all} r \geq 0. \tag{4.4}$$

In particular, we see that

$$\mathcal{D}_{i,n}^{(m)} \stackrel{d}{=} X_{m:n-i}. \tag{4.5}$$

Remark 4.1, *Note that (4.5) can also be proved as follows.*

The binomial expansion of $\left(e^{\lambda r} - 1\right)^{m-1}$ is

$$\left(e^{\lambda r} - 1\right)^{m-1} = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \left(e^{\lambda r}\right)^{k}. \tag{4.6}$$

Using (4.6), Equation (2.5) simplifies to

$$\begin{aligned} f_{\mathcal{D}_{i,n}^{(m)}}(r) &= m\lambda\left(e^{\lambda r} - 1\right)^{m-1} \binom{n-i}{m} e^{-(n-i)\lambda r} \\ &= m\lambda \binom{n-i}{m} \left(1 - e^{-\lambda r}\right)^{m-1} e^{-(n-i-m+1)\lambda r}, \ r > 0. \end{aligned} \tag{4.7}$$

Note that the pdf in (4.7) is the same as the pdf of the mth order statistics corresponding to a random sample of size n from the exponential distribution with parameter λ.

The identity (4.4) is the same result obtained by [2, 3], [9], and other authors.

Remark 4.2 *Equation (15) means that the spacings of the same order are identically distributed, regardless of the sample size, if they are at the same distance from the largest order statistic of the corresponding sample.*

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