Available online www.jsaer.com

Journal of Scientific and Engineering Research, 2018, 5(8):252-266

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

A Generalized Transmuted Gompertz-Makeham Distribution

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Abstract This paper introduces a generalized model of the transmuted Gompertz-Makeham distributions from which some known extensions of the Gompertz-Makeham distribution can be derived. The proposed generalized model is obtained by adding two extra parameters to the Gompertz-Makeham distribution so that it becomes more flexible and adequately able to describe the intricacy of the data. We will study many of the statistical properties of the generalized model, especially its moments, moment-generating function, quantile function, entropy, order statistics, moments of order statistics, and weighted probability moments.

Keywords Gompertz-Makeham Distribution, entropy, order statistics, moments of order statistics

1. Introduction

The Gompertz-Makeham (GM) distribution was introduced by Makeham [19] in 1860. It is an extended model of the Gompertz probability distribution that was introduced by Gompertz [10] in 1825. The GM distribution is a continuous probability distribution that has been widely used in survival analysis, modeling human mortality, constructing actuarial tables and growth models. It has been recently used in many fields of sciences including actuaries, biology, demography, gerontology, and computer science.

A comprehensive review of the history and theory of the GM probability distribution can be found in Marshall and Olkin [20]. Golubev [9] emphasizes the practical importance of this probability distribution. Detailed information about the GM distribution, its mathematical and statistical properties, and its applications can be found in Johnson et al. [15] and Dey et al. [6].

A random variable X is said to have a GM distribution with positive parameters α, β and γ if its cumulative distribution function (cdf) is given by

$$F_{GM}(x;\alpha,\beta,\gamma) = 1 - \exp\left\{(\alpha/\beta)\left(1-e^{\beta x}\right) - \gamma x\right\}, \quad x > 0. \tag{1.1}$$

The corresponding probability density function (pdf) is given as

$$f_{GM}(x;\alpha,\beta,\gamma) = \left(\gamma + \alpha e^{\beta x}\right)\exp\left\{(\alpha/\beta)\left(1-e^{\beta x}\right) - \gamma x\right\}, \quad x > 0. \tag{1.2}$$

In statistics, it is always desired to extend classical distributions and generate new flexible distributions in order to adequately fit real lifetime date. The literature is rich of studies that aim at proposing methods to generate and extend new families of univariate continuous probability models (see, for example, Lee et al. [18]). Among those methods that have recently attracted statisticians and gained their attention is transmutation.

Transmuted distributions were introduced by Shaw and Buckley [24] in 2009 to extend known non-Gaussian distributions by adding extra parameters to their distribution functions. Transmuted distributions provide statisticians with tools to control the skewness and kurtosis of the distribution in order to fit their real data.

Given a baseline probability distribution with cdf $G(x)$ and pdf $g(x)$, a random variable X is said to have a transmutation map of quadratic rank if its cdf $F(x)$ and pdf $f(x)$ have the following simple forms

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x),$$
$$f(x) = g(x)[1+\lambda - 2\lambda G(x)], \quad |\lambda| \le 1.$$

Recently, many authors have proposed methods to extend the GM model. Abdul-Moniem and Seham [1] introduced the transmuted Gompertz distribution and studied its statistical properties. khan et al. [16] introduced a new three parameter aging distribution which is a generalization of the gompertz distribution and studied its properties. El-Gohary et al. [8] proposed a generalized Gompertz distribution with cdf and pdf given respectively by

$$F(x;\alpha,\beta,\eta) = \left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta}, \quad x>0,$$

$$f(x;\alpha,\beta,\eta) = \alpha\beta e^{\eta x}\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta-1}, \quad x>0,$$

where α, η>0 are the scale parameters and β >0 is the shape parameter.

In 2017, Khan et al. [17] introduced the four parameter transmuted generalized Gompertz distribution with a cdf and pdf given respectively by

$$F(x;\alpha,\beta,\eta) = \left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta}\left\{1+\lambda-\lambda\left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta}\right\},$$

$$f(x;\alpha,\beta,\eta) = \alpha\beta e^{\eta x}\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta-1}$$
$$\times\left\{1+\lambda-2\lambda\left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta}\right\},$$

where $\alpha,\beta,\eta>0, \lambda\le 1$, and $x>0$.

El-Bar [7] introduced an extended Gompertz-Makeham model and studied its properties. It is in fact a transmuted Gompertz-Makeham (TGM) distribution that has a cdf

$$F_{TGM}(x;\alpha,\beta,\gamma) = \left\{1-e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x}\right\}\left\{1+\lambda e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x}\right\}, \quad (1.3)$$

where $\alpha>0, \beta>0, \gamma>0$, and $|\lambda|\le 1$.

The corresponding pdf is given by

$$f_{TGM}(x;\alpha,\beta,\gamma) = \left(\gamma+\alpha e^{\beta x}\right)e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x}$$
$$\times\left\{1-\lambda+2\lambda e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x}\right\}, \quad x>0. \quad (1.4)$$

In this paper, we would like to generalize the TGM distribution by adding two more parameters to its distribution functions. The cdf, for $x > 0$, of the proposed model is

$$F_X(x;\Omega) = 1 - \left[e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x} \right]^{\delta} \left\{ 1 - \lambda + \lambda \left[e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x} \right]^{\varepsilon} \right\},$$

where Ω is the vector $(\alpha, \beta, \gamma, \delta, \varepsilon, \lambda)$, $\delta \geq 1, \varepsilon > 0, \alpha > 0, \beta > 0, \gamma > 0$, and $|\lambda| \leq 1$.

The corresponding pdf, for $x > 0$, is

$$f_X(x;\Omega) = \left(\gamma + \alpha e^{\beta x}\right) \left[e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x} \right]^{\delta} \left\{ \delta(1-\lambda) + \lambda(\delta+\varepsilon) \left[e^{(\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma x} \right]^{\varepsilon} \right\}.$$

2. Generalized Transmuted GM Distribution

We introduce the generalized transmuted Gompertz-Makeham (GTGM) distribution. The derivation of the GTGM distribution is based on a higher rank transmuted (HRT-G) family of distributions introduced in Riffi [23]. The cdf of the HRT-G distribution for some baseline distribution with cdf $G(x)$ is given by

$$F_k(x) = 1 - (1-G(x))^{\alpha_1} \left\{ 1 - \lambda_1 + \sum_{j=2}^{k-1} \left(\lambda_{j-1} - \lambda_j\right)(1-G(x))^{\alpha_j} + \lambda_{k-1}(1-G(x))^{\alpha_k} \right\}, \tag{2.1}$$

where $\alpha_1 \geq 1$, $\alpha_{j+1} \geq 0$, and $0 \leq \lambda_1 \leq 1$, $\lambda_j - 1 \leq \lambda_{j+1} \leq \lambda_j$ for $j = 1, \ldots, k-1$.

When $k = 2, \alpha_1 = \delta, \alpha_2 = \varepsilon$, and $G(x)$ is the cdf of the Gompertz-Makeham distribution, the cdf of the resulting distribution is given by

$$F_X(x;\Omega) = 1 - e^{(\alpha/\beta)\delta\left(1-e^{\beta x}\right)-\gamma\delta x} \left\{ 1 - \lambda + \lambda e^{(\alpha/\beta)\varepsilon\left(1-e^{\beta x}\right)-\gamma\varepsilon x} \right\}. \tag{2.2}$$

The pdf of the GTGM distribution is given by

$$f_X(x;\Omega) = \left(\gamma + \alpha e^{\beta x}\right) e^{(\delta/\beta)\alpha\left(\left(1-e^{\beta x}\right)-\alpha\gamma x\right)} \left\{ \delta(1-\lambda) + \lambda(\delta+\varepsilon) e^{(\alpha/\beta)\varepsilon\left(1-e^{\beta x}\right)-\gamma\varepsilon x} \right\}. \tag{2.3}$$

The hazard rate function of X is given, for $x > 0$, as

$$h_X(x) = (\gamma + \alpha e^{\beta x}) \left\{ \delta(1-\lambda) + (\delta+\varepsilon)\lambda e^{\frac{\alpha\varepsilon\left(1-e^{\beta x}\right)}{\beta} - \gamma x\varepsilon} \right\} / \left\{ 1 - \lambda + \lambda e^{\frac{\alpha\varepsilon\left(1-e^{\beta x}\right)}{\beta} - \gamma x\varepsilon} \right\}.$$

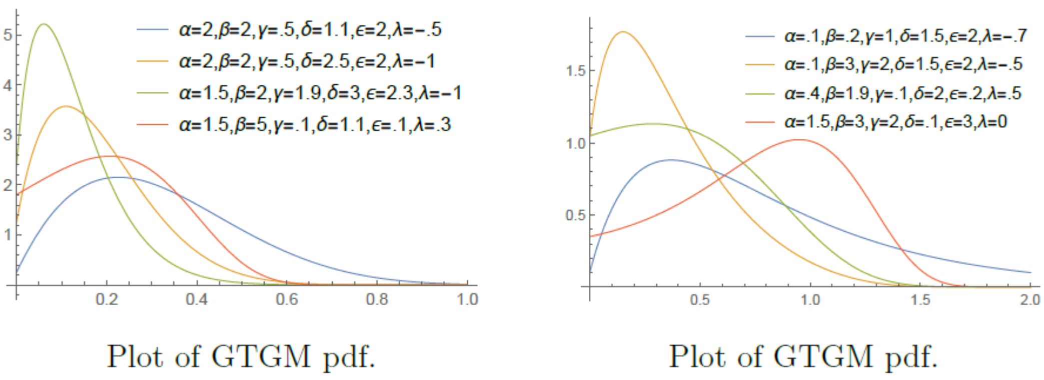

Figure 1: Plot of the GTGM pdf for a variety of values of its parameters

The GTGM model generalizes an extended Gompertz-Makeham (TGM) distribution studied by El-Bar [7]. In fact, we derive the TGM distribution from the GTGM distribution by letting $\delta = \varepsilon = 1$.
The expression "generalized" in the title of this paper comes from the fact that the cdf of the GTGM distribution can be written as

$$F_X(x;\Omega) = 1 - \sum_{i=0}^{\infty}(-1)^i \left\{(1-\lambda)\binom{\delta}{i} + \lambda\binom{\delta+\varepsilon}{i}\right\}G(x)^i$$

$$= \sum_{i=1}^{\infty}(-1)^{i+1} \left\{(1-\lambda)\binom{\delta}{i} + \lambda\binom{\delta+\varepsilon}{i}\right\}G(x)^i, \quad (2.4)$$

where $G(x)$ is the cdf of the GM distribution.
Note that (2.4) can be justified by using the general binomial theorem.
Similarly, the pdf of the GTGM distribution can be written as

$$f_X(x;\Omega) = g(x)\sum_{i=1}^{\infty}(-1)^{i+1} i \left\{(1-\lambda)\binom{\delta}{i} + \lambda\binom{\delta+\varepsilon}{i}\right\}G(x)^{i-1}. \quad (2.5)$$

For example, if $\delta = \varepsilon = 1$, then

$$F(x) = (1+2\lambda)G(x) - 3\lambda G(x)^2 + \lambda G(x)^3 \quad (2.6)$$

is the cdf of a special case of the cubic rank transmuted GM distribution introduced by Granzotto et al. [11] (when $\lambda_1 = 1+2\lambda$ and $\lambda_2 = 1-\lambda$ in (3) of [11]).
In the sequel, we will say that a random variable X has a GTGM distribution with parameters $\alpha, \beta, \gamma, \delta, \varepsilon$, and λ, abbreviated as $X : GTGM(\Omega)$, if its cdf is given by (2.2), where Ω is the vector $(\alpha, \beta, \gamma, \delta, \varepsilon, \lambda)$, $\delta \geq 1, \varepsilon > 0, \alpha > 0, \beta > 0, \gamma > 0$, and $|\lambda| \leq 1$. We will let $F_X(x)$ and $f_X(x)$ denote the cdf and the pdf of $X : GTGM(\Omega)$, respectively.

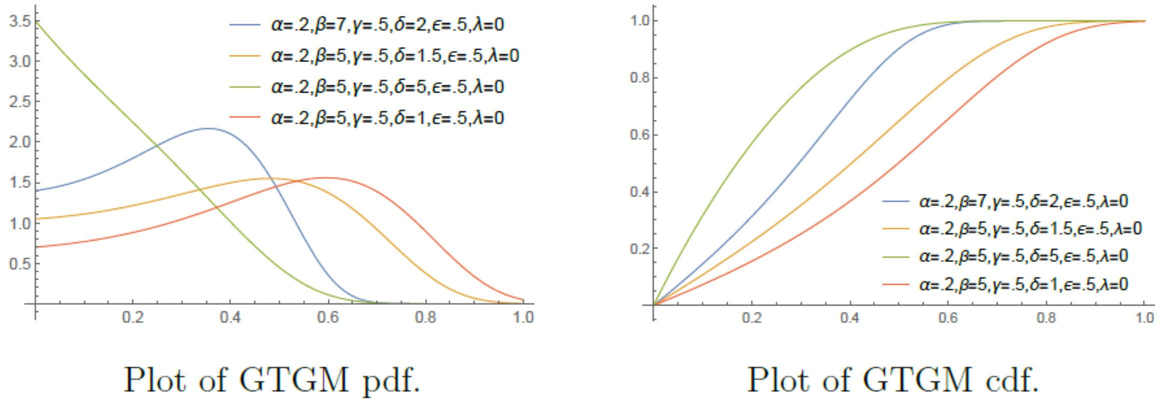

Figure 2: Plot of the GTGM pdf and cdf for a variety of values of its parameters

3. Sub-models and Possible Extension

1. If we let $\delta = \varepsilon = 1$ in (2.2), then we get the quadratic transmuted GM distribution described in (1.3).
2. If we let $\delta = \varepsilon = 1$ and $\lambda = 0$ in (2.2), then we get the standard GM distribution with parameters α, β, and γ.
3. If we let $\delta = \varepsilon = 1$ and $\alpha = \lambda = 0$ in (2.2), then we get the exponential distribution with parameter γ.

4. If we let $\delta = \varepsilon = 1$ and $\gamma = 0$ in (2.2), then we get the cubic transmuted Gompertz distribution described in (2.6).

It is possible to extend the GTGM model by exponentiated the GM distribution that we use as a baseline; i.e., we replace $G(x)$ by $G(x)^a$ in (2.4), where $a > 0$. That is, the cdf of the extended GTGM model will be

$$F_E(x) = 1 - \left[1 - G(x)^a\right]^{\delta}\left\{1 - \lambda + \lambda\left[1 - G(x)^a\right]^{\varepsilon}\right\} \quad (3.1)$$

Many known models will be special sub-models of the extended GTGM model.

For example, with $\delta = \varepsilon = 1$, we get the cubic transmuted GM model proposed by Aslam et al. [3] with $\lambda_1 = 1 + 2\lambda$ and $\lambda_2 = 1 - \lambda$, namely

$$F(x) = \lambda_1 G(x)^a + (\lambda_2 - \lambda_1)G(x)^{2a} + (1 - \lambda_2)G(x)^{3a}. \quad (3.2)$$

The Kumaraswamy GM distribution introduced by Chukwu and Ogunde [4] can also be derived from the extended GTGM model. In fact, if we let $\delta = 1, \varepsilon = b - 1$, and $\lambda = 1$ in (2.4), the cdf of the generated distribution will be

$$\begin{aligned} F(x;\alpha,\beta,\gamma,a,b) &= 1 - \left[1 - G(x)^a\right]\left[1 - G(x)^a\right]^{b-1} \\ &= 1 - \left[1 - G(x)^a\right]^b = 1 - \left\{1 - \left(1 - e^{(\alpha/\beta)(1 - e^{\beta x}) - \gamma x}\right)^a\right\}^b. \end{aligned} \quad (3.3)$$

It is also remarkable to mention that the generalized transmuted Gompertz-Makeham model introduced by Alizadeh et al. [2] is a special case of extended model with cdf given by (3.1). To see this, let $\delta = \varepsilon = 1$ in (3.1). Then, the cdf of the generated model will be

$$F(x) = 1 - \left\{1 - \lambda G(x)^a\right\}\left\{1 - G(x)^a\right\}, \quad a > 0, |\lambda| \le 1.$$

4. GTGM as Mixture of Distributions

The GTGM distribution is a mixture of two GM distributions with weights $1 - \lambda$ and λ as in the described in the following equation.

$$f_X(x;\Omega) = (1 - \lambda) f_1(x;\Omega) + \lambda f_2(x;\Omega),$$

where the functions $f_1(x;\Omega)$ and $f_2(x;\Omega)$ are given by

$$f_1(x;\Omega) = \delta\left(\gamma + \alpha e^{\beta x}\right)\exp\left\{(\alpha/\beta)\delta\left(1 - e^{\beta x}\right) - \gamma\delta x\right\} \quad (4.1)$$

$$f_2(x;\Omega) = (\delta + \varepsilon)\left(\gamma + \alpha e^{\beta x}\right)\exp\left\{(\alpha/\beta)(\delta + \varepsilon)\left(1 - e^{\beta x}\right) - \gamma(\delta + \varepsilon)x\right\}. \quad (4.2)$$

Here, $f_1(x)$ is the pdf of a GM random variable with parameters $\beta, \delta\alpha$, and $\delta\gamma$. Similarly, $f_2(x)$ is the pdf of a GM random variable with parameters $\beta, (\delta + \varepsilon)\alpha$, and $(\delta + \varepsilon)\gamma$.

5. Moment-Generating Function

The moment-generating function (mgf) of the GTGM distribution can be calculated using the transformation of variables technique. The result will be given in terms of the generalized integro-exponential function which is defined by

$$E^r_{(s)}(z) = \frac{1}{\Gamma(r+1)}\int_1^{\infty}(\log u)^r u^{-s} e^{-zu}\,du, z > 0. \quad (5.1)$$

As a special case, $E^0_{(s)}(z)$ is the exponential integral given as

$$E_s(z) = \int_1^{\infty} u^{-s} e^{-zu}\,du. \quad (5.2)$$

Below, we are going to use the following identity from [21].

$$E_{(s)}^{r-1}(z) = (s-1)E_{(s)}^{r}(z) + zE_{(s-1)}^{r}(z), \quad z > 0. \tag{5.3}$$

Theorem 5.1 *Let* $X : GTGM(\Omega)$. *Then the moment-generating function of* X *is given by*

$$M_X(t) = (1-\lambda)\delta e^{\frac{\alpha\delta}{\beta}}\left\{\frac{\alpha}{\beta}E^{0}_{\left(\frac{\gamma\delta-t}{\beta}\right)}\left(\frac{\alpha\delta}{\beta}\right) + \frac{\gamma}{\beta}E^{0}_{\left(\frac{\gamma\delta-t}{\beta}+1\right)}\left(\frac{\alpha\delta}{\beta}\right)\right\}$$
$$+\lambda(\delta+\varepsilon)e^{\frac{\alpha(\delta+\varepsilon)}{\beta}}\left\{\frac{\alpha}{\beta}E^{0}_{\left(\frac{\gamma(\delta+\varepsilon)-t}{\beta}\right)}\left(\frac{\alpha(\delta+\varepsilon)}{\beta}\right) + \frac{\gamma}{\beta}E^{0}_{\left(\frac{\gamma(\delta+\varepsilon)-t}{\beta}+1\right)}\left(\frac{\alpha(\delta+\varepsilon)}{\beta}\right)\right\}. \tag{5.4}$$

Proof. Let $Y = e^{\beta X}$. Then the pdf of Y is

$$f_Y(y;\alpha,\beta,\gamma,\tau) = \sum_{i=1}^{2}\lambda_i\tau_i(\gamma+\alpha y)\beta^{-1}e^{-\frac{\alpha\tau_i(y-1)}{\beta}}y^{-\frac{\gamma\tau_i}{\beta}-1} \qquad y > 1, \tag{5.5}$$

where $\tau_1 = \delta, \tau_2 = \delta+\varepsilon, \lambda_1 = 1-\lambda$, and $\lambda_2 = \lambda$.

Now, the mgf of X is given by

$$M_X(t) = E\left(e^{tX}\right) = E\left(Y^{t/\beta}\right)$$
$$= \sum_{i=1}^{2}\lambda_i\beta^{-1}\int_1^{\infty}\tau_i y^{t/\beta}e^{-\frac{\alpha\tau_i(y-1)}{\beta}}y^{-\frac{\gamma\tau_i}{\beta}-1}(\gamma+\alpha y)dy. \tag{5.6}$$

Then, in terms of the generalized integro-exponential function, $E_{(s)}^{r}(z)$, the mgf of X can be written as

$$M_X(t) = (1-\lambda)\delta e^{\frac{\alpha\delta}{\beta}}\left\{\frac{\alpha}{\beta}E^{0}_{\left(\frac{\gamma\delta-t}{\beta}\right)}\left(\frac{\alpha\delta}{\beta}\right) + \frac{\gamma}{\beta}E^{0}_{\left(\frac{\gamma\delta-t}{\beta}+1\right)}\left(\frac{\alpha\delta}{\beta}\right)\right\}$$
$$+\lambda(\delta+\varepsilon)e^{\frac{\alpha(\delta+\varepsilon)}{\beta}}\left\{\frac{\alpha}{\beta}E^{0}_{\left(\frac{\gamma(\delta+\varepsilon)-t}{\beta}\right)}\left(\frac{\alpha(\delta+\varepsilon)}{\beta}\right) + \frac{\gamma}{\beta}E^{0}_{\left(\frac{\gamma(\delta+\varepsilon)-t}{\beta}+1\right)}\left(\frac{\alpha(\delta+\varepsilon)}{\beta}\right)\right\}.$$

Corollary 5.1 *The* r *th partial derivative of* $M_X(t)$ *with respect to* t *is*

$$M_X^{(r)}(t) = \sum_{i=1}^{2}\lambda_i\beta^{-r}\Gamma(r+1)e^{\frac{\alpha\tau_i}{\beta}}\left\{(\gamma\tau_i/\beta)E^{r}_{\left(\frac{\gamma\tau_i-t}{\beta}+1\right)}\left(\frac{\alpha\tau_i}{\beta}\right) + (\alpha\tau_i/\beta)E^{r}_{\left(\frac{\gamma\tau_i-t}{\beta}\right)}\left(\frac{\alpha\tau_i}{\beta}\right)\right\}$$
$$= \sum_{i=1}^{2}\lambda_i\beta^{-r}\Gamma(r+1)e^{\frac{\alpha\tau_i}{\beta}}E^{r-1}_{\left(\frac{\gamma\tau_i}{\beta}+1\right)}\left(\frac{\alpha\tau_i}{\beta}\right). \tag{5.7}$$

Proof. To find the r th partial derivative of $M_X(t)$ with respect to t, we differentiate under the integral sine of the right-hand side of (5.6) to get

$$\frac{\partial^r}{\partial t^r}\left(M_X(t)\right) = \gamma\lambda_i\tau_i\beta^{-r-1}e^{\frac{\alpha\tau_i}{\beta}}\left(\int_1^{\infty}(\log y)^r e^{-\frac{\alpha\tau_y}{\beta}}y^{-\frac{\gamma\tau_i-t}{\beta}-1}dy\right) \tag{5.8}$$

$$+\alpha\lambda_i\tau_i\beta^{-r-1}e^{\frac{\alpha\tau_i}{\beta}}\left(\int_1^\infty(\log y)^r e^{-\frac{\alpha\tau_i y}{\beta}} y^{-\frac{\gamma\tau_i-t}{\beta}}dy\right). \tag{5.9}$$

In terms of the generalized integro-exponential function,

$$M_X^{(r)}(t)=(1-\lambda)m_1(t)+\lambda m_2(t),$$

where

$$m_i(t)=\beta^{-r}\Gamma(r+1)\lambda_i e^{\frac{\alpha\tau_i}{\beta}}\left\{(\gamma\tau_i/\beta)E^r_{\left(\frac{\gamma\tau_i-t}{\beta}+1\right)}\left(\frac{\alpha\tau_i}{\beta}\right)+(\alpha\tau_i/\beta)E^r_{\left(\frac{\gamma\tau_i-t}{\beta}\right)}\left(\frac{\alpha\tau_i}{\beta}\right)\right\} \tag{5.10}$$

and, as before, $\tau_1=\delta,\tau_2=\delta+\varepsilon,\lambda_1=1-\lambda$, and $\lambda_2=\lambda$.

Remark 5.1 *By letting* $t=0$ *in (5.10) and using the identity 5.3, we get the* r *th moment of* X *as in the following equation.*

$$E(X^r)=(1-\lambda)m_1(0)+\lambda m_2(0)$$

$$=\beta^{-r}\Gamma(r+1)e^{\frac{\alpha\delta}{\beta}}\left\{(1-\lambda)E^{r-1}_{\left(\frac{\gamma\delta}{\beta}+1\right)}\left(\frac{\alpha\delta}{\beta}\right)+\lambda e^{\frac{\alpha\varepsilon}{\beta}}E^{r-1}_{\left(\frac{\gamma(\delta+\varepsilon)}{\beta}+1\right)}\left(\frac{\alpha(\delta+\varepsilon)}{\beta}\right)\right\}.$$

6. Moments and Quantile Function

6.1. Moments

Theorem 6.1 *Let* $X: GTGM(\Omega)$. *Then, the* r *th moment of the* X *is given by*

$$E[X^r]=\beta^{-r}\Gamma(r+1)e^{\frac{\alpha\delta}{\beta}}\left\{(1-\lambda)E^{r-1}_{\left(\frac{\gamma\delta}{\beta}+1\right)}\left(\frac{\alpha\delta}{\beta}\right)+\lambda e^{\frac{\alpha\varepsilon}{\beta}}E^{r-1}_{\left(\frac{\gamma(\delta+\varepsilon)}{\beta}+1\right)}\left(\frac{\alpha(\delta+\varepsilon)}{\beta}\right)\right\}. \tag{6.1}$$

Proof. The r th moment of X is given by

$$E(X^r)=E(\beta^{-r}\log^r(Y)),$$

where $Y=e^{\beta X}$ has the pdf given by (5.5).

By the same technique we used to calculate the mgf of X, we see that

$$E(\beta^{-r}(\log Y)^r)=\sum_{i=1}^{2}\lambda_i\beta^{-1}\int_1^\infty\tau_i\beta^{-r}(\log y)^r(\gamma+\alpha y)e^{-\frac{\alpha\tau_i(y-1)}{\beta}}y^{-\frac{\gamma\tau_i}{\beta}-1}dy \tag{6.2}$$

$$=\sum_{i=1}^{2}\lambda_i\tau_i\beta^{-r}e^{\frac{\alpha\tau}{\beta}}\left\{\frac{\gamma}{\beta}\int_1^\infty(\log y)^r e^{-\frac{\alpha\tau y}{\beta}}y^{-\left(\frac{\gamma\tau}{\beta}+1\right)}dy\right. \tag{6.3}$$

$$\left.+\frac{\alpha}{\beta}\int_1^\infty(\log y)^r e^{-\frac{\alpha\tau_i y}{\beta}}y^{-\frac{\gamma\tau_i}{\beta}}dy\right\}. \tag{6.4}$$

Using the generalized integro-exponential function, we write

$$E(\beta^{-r}(\log Y)^r)=\sum_{i=1}^{2}\lambda_i\beta^{-r}r!e^{\frac{\alpha\tau_i}{\beta}}\left\{\frac{(\gamma\tau_i)}{\beta}E^r_{\left(\frac{\gamma\tau_i}{\beta}+1\right)}\left(\frac{\alpha\tau_i}{\beta}\right)+\frac{(\alpha\tau_i)}{\beta}E^r_{\left(\frac{\gamma\tau_i}{\beta}\right)}\left(\frac{\alpha\tau_i}{\beta}\right)\right\}$$

$$=\sum_{i=1}^{2}\lambda_i\beta^{-r}\Gamma(r+1)e^{\frac{\alpha\tau_i}{\beta}}E_{\left(\frac{\gamma\tau_i}{\beta}+1\right)}^{r-1}\left(\frac{\alpha\tau_i}{\beta}\right), \tag{6.5}$$

where $\tau_1=\delta,\tau_2=(\delta+\varepsilon),\lambda_1=1-\lambda,$ and $\lambda_2=\lambda$.

Hence, (6.1) follows.

6.2. Quantile Function

Theorem 6.2. *Let* $X : GTGM(\Omega)$ *with* $\varepsilon=\delta$, *for simplicity. Then the quantile function of* X *is given by*

$$x_q=[\alpha\delta-\beta\log(B(q,\lambda))]/(\beta\gamma\delta)-\beta^{-1}p\left(\alpha\gamma^{-1}e^{\alpha/\gamma}B(q,\lambda)^{-\frac{\beta}{\gamma\delta}}\right), \tag{6.6}$$

where $p(z)$ is the principal solution for w in $z=we^w$.

Proof. Assume that $\varepsilon=\delta$. To compute the quantile function x_q of X, we solve the following equation for x_q

$$1-\lambda+\lambda e^{\frac{\alpha\delta\left(1-e^{\beta x_q}\right)}{\beta}-\gamma\delta x_q}=(1-q)e^{\gamma\delta x_q-\frac{\alpha\delta\left(1-e^{\beta x_q}\right)}{\beta}}.$$

Let $y=\exp\{\frac{\alpha\delta\left(1-e^{\beta x_q}\right)}{\beta}-\gamma\delta x_q\}$. Then, the solution of the equation $1-\lambda+\lambda y^{\delta}=(1-q)y^{-\delta}$ is given as

$$y=e^{\frac{\alpha\left(1-e^{\beta x_q}\right)}{\beta}+\gamma x_q}=\left(\frac{-1+\lambda+\sqrt{1+2\lambda-4\lambda q+\lambda^2}}{2\lambda}\right)^{1/\delta}. \tag{6.7}$$

Now, let the function $B(q,\lambda)$ be defined by

$$B(q,\lambda)=\frac{-1+\lambda+\sqrt{1+2\lambda-4\lambda q+\lambda^2}}{2\lambda}.$$

Then, the solution of (6.7) reduces as required to

$$x_q=[\alpha\delta-\beta\log(B(q,\lambda))]/(\beta\gamma\delta)-\beta^{-1}p\left(\alpha\gamma^{-1}e^{\alpha/\gamma}B(q,\lambda)^{-\frac{\beta}{\gamma\delta}}\right),$$

where $p(z)$ is the principal solution for w in $z=we^w$ (or the Lambert W-function).

7. Probability Weighted Moments

Given a random variable X with a cumulative distribution function G, the probability weighted moments are defined to be

$$M(p,r,s)=E\left[X^pG(x)^r(1-G(x))^s\right],$$

where p,r, and s are all real numbers (see Hosking [14] and Greenwood et al. [12]).

A special case is when $p=1$, and $s=0$, namely

$$\beta_r=M(1,0,r)=E\left(XG(x)^r\right)$$

$$E\left(X^p\right)=\delta(1-\lambda)M(p,0,\delta-1)+\lambda(\delta+\varepsilon)M(p,0,\delta+\varepsilon-1)$$

The L-moments are defined as

$$\lambda_r = E\left(XG(x)P^*{}_{r-1}\right)$$

with P^*_{r-1} denoting the r th shifted Legendre ploynomials.

$$P^*(n,x) := P_n(2x-1).$$

8. Entropy

8.1. Rényi Entropy

The entropy of a random variable X with pdf $f(x)$ is a measure of variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Rényi entropy (Rényi [22]) is defined as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^\infty f(x)^\rho dx\right\}$$

where $\rho > 0$ and $\rho \neq 1$.

To compute $I_R(\rho)$, we consider the following three cases.

Case 1: $\lambda = 1$

We simplify the integral $\int_0^\infty f(x)^\rho dx$ by letting $y = e^{\beta x}$. Hence,

$$\int_0^\infty f_X(x;\Omega)^\rho dx = \beta^{-1}\int_1^\infty (\delta+\varepsilon)^\rho (\gamma+\alpha y)^\rho e^{-\frac{\alpha\rho(y-1)(\delta+\varepsilon)}{\beta}} y^{-\frac{\gamma\rho(\delta+\varepsilon)}{\beta}-1} dy.$$

Using the general binomial expansion

$$(\gamma+\alpha y)^\rho = \sum_{k=0}^\infty \alpha^k y^k \binom{\rho}{k}\gamma^{\rho-k}, \tag{8.1}$$

we see that

$$\int_0^\infty f_X(x;\Omega)^\rho dx = \beta^{-1}(\delta+\varepsilon)^\rho e^{\frac{\alpha\rho(\delta+\varepsilon)}{\beta}} \sum_{k=0}^\infty \alpha^k \binom{\rho}{k}\gamma^{\rho-k}\int_1^\infty e^{-\frac{\alpha\rho y(\delta+\varepsilon)}{\beta}} y^{-\frac{\gamma\rho(\delta+\varepsilon)}{\beta}+k-1} dy.$$

The above integral reduces to

$$\int_0^\infty f_X(x;\Omega)^\rho dx = \beta^{-1}(\delta+\varepsilon)^\rho e^{\frac{\alpha\rho(\delta+\varepsilon)}{\beta}} \sum_{k=0}^\infty \alpha^k \binom{\rho}{k}\gamma^{\rho-k} E_{\left(\frac{\gamma\rho(\delta+\varepsilon)}{\beta}-k+1\right)}\left(\frac{\alpha\rho(\delta+\varepsilon)}{\beta}\right),$$

where $E_n(z)$ is the exponential integral function given by (5.2).

Therefore, the Rényi entropy of X is

$$I_R(\rho) = \frac{1}{1-\rho}\log\left[(\delta+\varepsilon)^\rho \beta^{-1} e^{\frac{\alpha\rho(\delta+\varepsilon)}{\beta}} \sum_{k=0}^\infty \alpha^k \binom{\rho}{k}\gamma^{\rho-k} E_{\left(\frac{\gamma\rho(\delta+\varepsilon)}{\beta}-k+1\right)}\left(\frac{\alpha\rho(\delta+\varepsilon)}{\beta}\right)\right]$$

$$= \frac{1}{1-\rho}\left\{\frac{\alpha\rho(\delta+\varepsilon)}{\beta} - \log(\beta) + \rho\log(\delta+\varepsilon)\right.$$

$$\left. + \log\left[\sum_{k=0}^\infty \alpha^k \binom{\rho}{k}\gamma^{q-k} E_{\left(\frac{\gamma\rho(\delta+\varepsilon)}{\beta}-k+1\right)}\left(\frac{\alpha\rho(\delta+\varepsilon)}{\beta}\right)\right]\right\}. \tag{8.2}$$

Case 2: $\lambda = 0$

This case is similar to Case 2. The Rényi entropy of X is

$$I_R(\rho) = \frac{1}{1-\rho}\left\{\frac{\alpha\delta\rho}{\beta} - \log(\beta) + \rho\log(\delta) + \log\left[\sum_{k=0}^{\infty}\alpha^k\binom{\rho}{k}\gamma^{\rho-k}E_{\left(\frac{\gamma\delta\rho}{\beta}-k+1\right)}\left(\frac{\alpha\delta\rho}{\beta}\right)\right]\right\}.$$

Case 3: $\lambda \neq 0,1$

As in the above two cases, we can write $\int_0^{\infty} f(x)^{\rho}dx$ as

$$\int_0^{\infty} f_X(x;\Omega)^{\rho}dx = \beta^{-1}e^{\frac{\alpha\delta\rho}{\beta}}\int_1^{\infty}A(\rho)(\gamma+\alpha y)^{\rho}e^{-\frac{\alpha\delta\rho y}{\beta}}y^{-\frac{\gamma\delta\rho}{\beta}-1}dy,$$

where $A(\rho)$ is given by

$$A(\rho) = \left(\delta(1-\lambda) + \lambda(\delta+\varepsilon)e^{\frac{\alpha\varepsilon}{\beta}}e^{-\frac{\alpha y\varepsilon}{\beta}}y^{-\frac{\gamma\varepsilon}{\beta}}\right)^{\rho}$$

$$= \sum_{j=0}^{\infty}\lambda^j\binom{\rho}{j}\delta^{\rho-j}(\delta+\varepsilon)^j(1-\lambda)^{\rho-j}e^{\frac{\alpha j\varepsilon}{\beta}}e^{-\frac{\alpha jy\varepsilon}{\beta}}y^{-\frac{\gamma j\varepsilon}{\beta}}. \quad (8.3)$$

By using (8.1) and (5.2), the Rényi entropy can be written as

$$I_R(\rho) = \frac{1}{1-\rho}\left\{-\log(\beta) + \log\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda^j\alpha^k\binom{\rho}{j}\binom{\rho}{k}\delta^{\rho-j}(\delta+\varepsilon)^j(1-\lambda)^{\rho-j}\gamma^{\rho-k}\right.\right.$$

$$\left.\left.\times e^{\frac{\alpha(\delta\rho+j\varepsilon)}{\beta}}E_{\left(\frac{\beta+\gamma\delta\rho+\gamma j\varepsilon}{\beta}-k\right)}\left(\frac{\alpha(\delta\rho+j\varepsilon)}{\beta}\right)\right]\right\}. \quad (8.4)$$

8.2. q-Entropy

The q-entropy was introduced by Havrda and Charvat [13]. It is the one parameter generalization of the Shannon entropy. Ullah [25]defined the q-entropy as

$$I_H(q) = \frac{1}{q-1}\left\{1 - \int_0^{\infty} f(x)^q dx\right\}$$

where $q > 0$ and $q \neq 1$.

To compute $I_H(q)$, we consider the following three cases.

Case 1: $\lambda = 1$

By the same techniques we used in computing the Rényi entropy, we in this case that

$$I_H(q) = \frac{1}{q-1}\left\{1 - \beta^{-1}(\delta+\varepsilon)^q e^{\frac{\alpha q(\delta+\varepsilon)}{\beta}}\sum_{k=0}^{\infty}\alpha^k\binom{q}{k}\gamma^{q-k}\int_1^{\infty}e^{-\frac{\alpha qy(\delta+\varepsilon)}{\beta}}y^{k-\frac{\gamma q(\delta+\varepsilon)}{\beta}-1}dy\right\}$$

$$= \frac{1}{q-1}\left\{1-\beta^{-1}(\delta+\varepsilon)^{q} e^{\frac{\alpha q(\delta+\varepsilon)}{\beta}} \sum_{k=0}^{\infty}\alpha^{k}\binom{q}{k}\gamma^{q-k} E_{\left(-k+\frac{\gamma q(\delta+\varepsilon)}{\beta}+1\right)}\left(\frac{\alpha q(\delta+\varepsilon)}{\beta}\right)\right\}. \tag{8.5}$$

Case 2: $\lambda = 0$

In this case, the q -entropy is just simply,

$$I_H(q) = \frac{1}{q-1}\left\{1-\beta^{-1}\delta^{q} e^{\frac{\alpha\delta q}{\beta}} \sum_{k=0}^{\infty}\alpha^{k}\binom{q}{k}\gamma^{q-k} E_{\left(-k+\frac{\gamma\delta q}{\beta}+1\right)}\left(\frac{\alpha\delta q}{\beta}\right)\right\}.$$

Case 3: $\lambda \neq 0,1$

In this case, the q -entropy is given by

$$I_H q = \frac{1}{q-1}\left\{1-\beta^{-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda^{j}\alpha^{k}\binom{q}{j}\binom{q}{k}(\delta+\varepsilon)^{j}\delta^{q-j}(1-\lambda)^{q-j}\gamma^{q-k}\right.$$
$$\left.\times e^{\frac{\alpha(j\varepsilon+\delta q)}{\beta}} E_{\left(\frac{\beta+\gamma j\varepsilon+\gamma\delta q}{\beta}-k\right)}\left(\frac{\alpha(j\varepsilon+\delta q)}{\beta}\right)\right\}. \tag{8.6}$$

9. Order Statistics

let $X_1,\ldots,X_n$ be a random sample of size n from the GTGM distribution with parameters $\alpha > 0, \beta > 0$, and $\lambda > 0$, and let $X_{1:n}, X_{2:n},\ldots,X_{n:n}$ be the corresponding order statistics obtained by arranging $X_i, i = 1,\ldots,n$, in non-decreasing order of magnitude. The i th element of this sequence, $X_{i:n}$, is called the i th order statistic.

From DasGupta [5], the pdf of the i th order statistics is obtain from the equation

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!}\left[F_X(x)\right]^{i-1}\left[1-F_X(x)\right]^{n-i} f_X(x).$$

Therefore,

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!} w(\delta)(\delta(1-\lambda)+\lambda(\delta+\varepsilon)w(\varepsilon))\left(\gamma+\alpha e^{\beta x}\right) \tag{9.1}$$
$$\times\{1-(1-\lambda)w(\delta)-\lambda w(\delta+\varepsilon)\}^{i-1}\{(1-\lambda)w(\delta)+\lambda w(\delta+\varepsilon)\}^{n-i}, \tag{9.2}$$

where

$$w(\tau) = e^{(\tau\alpha/\beta)\left(1-e^{\beta x}\right)-\gamma\tau x}, \quad \tau > 0.$$

Below, we will compute the r th moment of the i th order statistics in three cases according to the value of λ.

Theorem 9.1 *Let $X_{i:n}$ be the i th order statistic from $X : GTGM(\Omega)$ with $\lambda = 0$. Then the r th moment of $X_{i:n}$ is given by*

$$E\left(X_{i:n}^{r}\right) = n\beta^{-r}\sum_{j=0}^{i-1}(-1)^{-j}\binom{i-1}{j}\binom{n-1}{i-1}\frac{\Gamma(r+1)}{c_{i,n}} e^{\frac{\alpha\delta c_{i,n}}{\beta}} E^{r-1}_{\left(\frac{\gamma\delta c_{i,n}}{\beta}+1\right)}\left(\frac{\alpha\delta c_{i,n}}{\beta}\right). \tag{9.3}$$

Proof. When $\lambda = 0$, the pdf of the i th order statistic is

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!}\delta w(\delta)(1-w(\delta))^{i-1} w(\delta)^{n-i}\left(\gamma+\alpha e^{\beta x}\right). \tag{9.4}$$

The pdf of the transformation $Y = e^{\beta X_{i:n}}$ for $y > 1$ is

$$f_Y(y) = n\delta\beta^{-1}\binom{n-1}{i-1} e^{-\frac{(y-1)(\alpha\delta)(n-i+1)}{\beta}} y^{-\frac{\gamma\delta(n-i+1)}{\beta}-1}\left\{1-e^{-\frac{\alpha\delta(y-1)}{\beta}} y^{-\frac{\gamma\delta}{\beta}}\right\}^{i-1}(\gamma+\alpha y)$$

$$= n\delta\beta^{-1}\sum_{j=0}^{i-1}(-1)^{-j}\binom{i-1}{j}\binom{n-1}{i-1} y^{-\frac{\gamma\delta c_{i,n}}{\beta}-1} e^{-\frac{\alpha\delta(y-1)c_{i,n}}{\beta}}(\gamma+\alpha y), \tag{9.5}$$

where $c_{i,n} = n-i+j+1$.

$$E\left(X_{i:n}^r\right) = E\left(\beta^{-r}(\log Y)^r\right) = \delta n\beta^{-r}\sum_{j=0}^{i-1}\beta^{-1}(-1)^{-j}\binom{i-1}{j}\binom{n-1}{i-1} e^{\frac{\alpha\delta c_{i,n}}{\beta}} I_1, \tag{9.6}$$

where

$$I_1 = \frac{1}{\beta}\int_1^\infty (\log y)^r(\gamma+\alpha y) e^{-\frac{\alpha\delta y c_{i,n}}{\beta}} y^{-\frac{\gamma\delta c_{i,n}}{\beta}-1} dy$$

$$= \frac{\alpha}{\beta}\int_1^\infty (\log y)^r e^{-\frac{\alpha\delta y c_{i,n}}{\beta}} y^{-\frac{\gamma\delta c_{i,n}}{\beta}} dy + \frac{\gamma}{\beta}\int_1^\infty \log^r(y) e^{-\frac{\alpha\delta y c_{i,n}}{\beta}} y^{-\frac{\gamma\delta c_{i,n}}{\beta}-1} dy$$

$$= \frac{\Gamma(r+1)}{\delta c_{i,n}}\left\{\frac{(\alpha\delta c_{i,n})}{\beta} E^r_{\left(\frac{\gamma\delta c_{i,n}}{\beta}\right)}\left(\frac{\alpha\delta c_{i,n}}{\beta}\right) + \frac{(\gamma\delta c_{i,n})}{\beta} E^r_{\left(\frac{\gamma\delta c_{i,n}}{\beta}+1\right)}\left(\frac{\alpha\delta c_{i,n}}{\beta}\right)\right\}$$

$$= \frac{\Gamma(r+1)}{\delta c_{i,n}} E^{r-1}_{\left(\frac{\gamma\delta c_{i,n}}{\beta}+1\right)}\left(\frac{\alpha\delta c_{i,n}}{\beta}\right). \tag{9.7}$$

Hence, the r th moment of $X_{i:n}$ is

$$E\left(X_{i:n}^r\right) = n\beta^{-r}\sum_{j=0}^{i-1}(-1)^{-j}\binom{i-1}{j}\binom{n-1}{i-1}\frac{\Gamma(r+1)}{c_{i,n}} e^{\frac{\alpha\delta c_{i,n}}{\beta}} E^{r-1}_{\left(\frac{\gamma\delta c_{i,n}}{\beta}+1\right)}\left(\frac{\alpha\delta c_{i,n}}{\beta}\right).$$

Theorem 9.2 *Let* $X_{i:n}$ *be the* i *th order statistic from* $X : GTGM(\Omega)$ *with* $\lambda = 1$. *Then the* r *th moment of* $X_{i:n}$ *is given by*

$$E\left(X_{i:n}^r\right) = n\beta^{-r}\sum_{j=0}^{i-1}(-1)^{-j}\binom{i-1}{j}\binom{n-1}{i-1}\frac{\Gamma(r+1)}{c_{i,n}}$$

$$\times e^{\frac{\alpha(\delta+\varepsilon)c_{i,n}}{\beta}} E^{r-1}_{\left(\frac{\gamma(\delta+\varepsilon)c_{i,n}}{\beta}+1\right)}\left(\frac{\alpha(\delta+\varepsilon)c_{i,n}}{\beta}\right). \tag{9.8}$$

Proof. When $\lambda = 1$, the pdf of $X_{i:n}$ becomes

$$f_{X_{i:n}}(x)=n(\delta+\varepsilon)\binom{n-1}{i-1}w((\delta+\varepsilon)(n-i+1))(1-w(\delta+\varepsilon))^{i-1}\left(\gamma+\alpha e^{\beta x}\right).$$

Then, we use the same technique as above in Theorem 9.1.

Theorem 9.3 *Let* $X_{i:n}$ *be the* i *th order statistic from* $X:GTGM(\Omega)$ *such that* $\lambda\neq 0$ *and* $\lambda\neq 1$. *Then the* r *th moment of* $X_{i:n}$ *is given by*

$$E\left(X_{i:n}^{r}\right)=n\beta^{-r}\binom{n-1}{i-1}\sum_{j=0}^{i-1}\sum_{k=0}^{j}\sum_{h=0}^{n-i}(-1)^{j}\binom{i-1}{j}\binom{j}{k}\binom{n-i}{h}\lambda^{h+k}(1-\lambda)^{-h-i+j-k+n}$$
$$\times e^{\frac{\alpha b_{i,n}^{1}}{\beta}}\left\{\frac{\delta(1-\lambda)}{b_{i,n}^{1}}E_{\left(\frac{\gamma b_{i,n}^{1}}{\beta}+1\right)}^{r-1}\left(\frac{\alpha b_{i,n}^{1}}{\beta}\right)+\frac{\lambda(\delta+\varepsilon)e^{\frac{\alpha\varepsilon}{\beta}}}{b_{i,n}^{2}}E_{\left(\frac{\gamma b^{2}_{i,n}}{\beta}+1\right)}^{r-1}\left(\frac{\alpha b_{i,n}^{2}}{\beta}\right)\right\}, \tag{9.9}$$

where $b_{i,n}^{1}=\delta(n-i+j+1)+(h+k)\varepsilon$ and $b_{i,n}^{2}=\delta(n-i+j+1)+(h+k+1)\varepsilon$.

Proof. By using the transformation $Y=e^{\beta X_{i:n}}$, we see that the pdf of Y at $y>1$ can be written as

$$f_{Y}(y)=\beta^{-1}n\binom{n-1}{i-1}(\gamma+\alpha y)e^{-\frac{\alpha\delta(y-1)}{\beta}}e^{-\frac{\alpha\delta(y-1)(n-i)}{\beta}}y^{-\frac{\gamma\delta(n-i+1)}{\beta}-1}$$
$$\times\left\{1-y^{-\frac{\gamma\delta}{\beta}}e^{-\frac{\alpha\delta(y-1)}{\beta}}\left[1-\lambda+\lambda e^{-\frac{\alpha\varepsilon(y-1)}{\beta}}y^{-\frac{\gamma\varepsilon}{\beta}}\right]\right\}^{i-1}$$
$$\times\left\{\delta(1-\lambda)+(\delta+\varepsilon)\lambda e^{-\frac{\alpha(y-1)\varepsilon}{\beta}}y^{-\frac{\gamma\varepsilon}{\beta}}\right\}\left\{1-\lambda+\lambda e^{-\frac{\alpha\varepsilon(y-1)}{\beta}}y^{-\frac{\gamma\varepsilon}{\beta}}\right\}^{n-i}. \tag{9.10}$$

Now, we use the following binomial expansions

$$\left\{1-y^{-\frac{\gamma\delta}{\beta}}e^{-\frac{\alpha\delta(y-1)}{\beta}}\left[1-\lambda+\lambda e^{-\frac{\alpha\varepsilon(y-1)}{\beta}}y^{-\frac{\gamma\varepsilon}{\beta}}\right]\right\}^{i-1}$$
$$=\sum_{j=0}^{i-1}\sum_{k=0}^{j}(-1)^{j}\lambda^{k}\binom{i-1}{j}\binom{j}{k}(1-\lambda)^{j-k}e^{-\frac{\alpha(y-1)(\delta j+k\varepsilon)}{\beta}}y^{-\frac{\gamma(\delta j+k\varepsilon)}{\beta}}\quad\text{and}$$

$$\left[1-\lambda+\lambda e^{-\frac{\alpha\varepsilon(y-1)}{\beta}}y^{-\frac{\gamma\varepsilon}{\beta}}\right]^{n-i}=\sum_{h=0}^{n-i}\lambda^{h}\binom{n-i}{h}(1-\lambda)^{n-i-h}e^{-\frac{h\alpha\varepsilon(y-1)}{\beta}}y^{-\frac{\gamma h\varepsilon}{\beta}}.$$

Therefore,

$$f_{Y}(y)=n\beta^{-1}\binom{n-1}{i-1}e^{-\frac{\alpha\delta(y-1)}{\beta}}y^{-\frac{\gamma(\delta+b_{i,n})}{\beta}-1}\left\{\delta(1-\lambda)+(\delta+\varepsilon)\lambda e^{-\frac{\alpha(y-1)\varepsilon}{\beta}}y^{-\frac{\gamma\varepsilon}{\beta}}\right\}(\gamma+\alpha y)$$

$$\times\sum_{j=0}^{i-1}\sum_{k=0}^{j}\sum_{h=0}^{n-i}(-1)^{j}\binom{i-1}{j}\binom{j}{k}\lambda^{h+k}\binom{n-i}{h}(1-\lambda)^{n-i+j-h-k}e^{-\frac{\alpha(y-1)b_{i,n}^{0}}{\beta}}, \tag{9.11}$$

where $b_{i,n}^{0}=\delta(n-i+j)+(h+k)\varepsilon$.

Using the same techniques as above in Theorem 9.3, we see that

$$E\left(X^{r}\right)=n\beta^{-r}\binom{n-1}{i-1}\sum_{j=0}^{i-1}\sum_{k=0}^{j}\sum_{h=0}^{n-i}(-1)^{j}\binom{i-1}{j}\binom{j}{k}\binom{n-i}{h}\lambda^{h+k}(1-\lambda)^{-h-i+j-k+n}$$

$$\times e^{\frac{\alpha b_{i,n}^{1}}{\beta}}\left\{\frac{\delta(1-\lambda)}{b_{i,n}^{1}}E_{\left(\frac{\gamma b_{i,n}^{1}}{\beta}+1\right)}^{r-1}\left(\frac{\alpha b_{i,n}^{1}}{\beta}\right)+\frac{\lambda(\delta+\varepsilon)e^{\frac{\alpha\varepsilon}{\beta}}}{b_{i,n}^{2}}E_{\left(\frac{\gamma b^{2}{}_{i,n}}{\beta}+1\right)}^{r-1}\left(\frac{\alpha b_{i,n}^{2}}{\beta}\right)\right\},$$

where $b_{i,n}^{1}=\delta(n-i+j+1)+(h+k)\varepsilon$ and $b_{i,n}^{2}=\delta(n-i+j+1)+(h+k+1)\varepsilon$.

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