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A Two-Dimensional Fractional Order Thermoelastic Problem for a Half-Space

S.E. Khader

Mathematical and Theoretical Physics Atomic Energy Authority, EGYPT

Abstract In this work, we apply the fractional order theory of thermoelasticity to a problem of a half-space. Whose surface is rigidly fixed and subjected to the effects of a thermal shock Laplace transform are using for the time variable and the exponential Fourier transform for one of the space variables. The solution in the transformed domain is obtained by a direct approach. Numerical inversion techniques are used to obtain the inverse double transforms. The temperature, thermal stresses, and the displacement distributions are obtained numerically for different values of the fractional parameter and different values of times. The results are represented graphically and discussed.

Keywords Fourier Transform; Fractional order thermoelasticity; Laplace transform; Rigidly Surface; Half-Space

Introduction

Thermoelasticity's importance is due to its many applications in diverse fields such as geophysics, plasma physics and related topics, especially in the design of nuclear reactors. There are several generalized theories of thermoelasticity. The first was developed by Lord and Shulman [1] and is often referred to as the L-S theory. In this theory the new equation of heat was obtained by using a new law of heat conduction instead of Fourier's law. Sherief [2] discussed the fundamental solution of the generalized thermoelastic problem for short times. Sherief and Anwar [3] solved a two dimensional generalized thermoelasticity problem for an infinitely long cylinder. Sherief and Helmy [4] are studied the solution of a two-dimensional generalized thermoelasticity problem for a half-space. Sherief and Megahed [5] are solved the problem of a two-dimensional thermoelasticity problem for a half- space subjected to heat sources, Sherief and Khader [6] are discussed the propagation of discontinuities in electromagneto generalized thermoelasticity in cylindrical regions. Sharma, Kumar and Chand [7] are discussed the Reflection of generalized thermoelastic waves from the boundary of a half-space. The second theory was developed by Green and Lindsay [8]. They introduced two different lag times in the stress- strain relations and the entropy expression. This theory is known as the theory of thermoelasticity with two relaxation times. Sherief and Megahed [9] solved a two dimensional problem for thermoelasticity with two relaxation times in spherical regions under axisymmetric distributions. Sherief [10-11] is discussed the fundamental solution for thermoelasticity with two relaxation times and a thermo-mechanical shock problem for thermoelasticity with two relaxation times.

The fractional order theory of thermoelasticity is based on the heat conduction equation with differential operators of fractional order. Time fractional differential operators describe memory effects, space fractional differential operator's deal with the long range interaction. The fractional calculus has been successfully used in engineering, physics, chemistry, geology, robotics bioengineering, robotics, etc. The first paper on fractional thermoelasticity was published by Povstenko [12] in 2005. Then after that published many research in this topic. Povstenko [13-15] proposed and investigated new models that use fractional derivative. The fractional order theory of thermoelasticity was derived by Sherief, El-Sayed, and Abd El-Latief [16]. Caputo and Mainardi [17-18] are discussed the new dissipation model based on memory mechanism, Pure and Applied Geophysics and

Linear model of dissipation in anelastic solids. Caputo [19] is studied the vibrations on an infinite viscoelastic layer with a dissipative memory. Abd El-Latief, and Khader [20] are solved the problem of Fractional model of thermoelasticity for a half-space overlaid by a thick layer.

Formulation of the problem

Consider a homogeneous isotropic elastic solid occupying the half-space $y \geq 0$. The y-axis is taken perpendicular to the plane. The surface of this medium is rigidly fixed and the surface is kept at a given temperature that is a function of *x*.

The displacement vector u has the form

$$\mathbf{u}=(u,v,0)$$

The cubical dilatation e is given by

$$e = \operatorname{div}\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{1}$$

The components of the thermoelastic stress tensor σ_{ij} are given by

$$\sigma_{xx} = (\lambda+2\mu)e - 2\mu\frac{\partial v}{\partial y} - \gamma(T-T_0)\,, \tag{2a}$$

$$\sigma_{yy} = (\lambda+2\mu)e - 2\mu\frac{\partial u}{\partial x} - \gamma(T-T_0)\,, \tag{2b}$$

$$\sigma_{zz} = \lambda e - \gamma(T-T_0)\,, \tag{2c}$$

$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{2d}$$

where λ and μ are Lamé's modulii, T is the absolute temperature of the medium, and γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$ where α_t is the coefficient of linear thermal expansion, T_0 is a reference temperature assumed to be such that $\left|(T-T_0)/T_0\right| << 1$.

The equations of motion have the vector form

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \gamma\operatorname{grad}T = \rho\,\ddot{\mathbf{u}}\ . \tag{3}$$

$$(\lambda+\mu)\frac{\partial e}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \gamma\frac{\partial T}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{4}$$

$$(\lambda+\mu)\frac{\partial e}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \gamma\frac{\partial T}{\partial y} = \rho\frac{\partial^2 v}{\partial t^2} \tag{5}$$

The equation of heat conduction has the form [16]

$$k\nabla^2 T = \frac{\partial}{\partial t}\left(1+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho c_E T + \gamma T_0\, e\right). \tag{6}$$

where k is the thermal conductivity of the medium, c_E is the specific heat at constant and ρ is the density. τ_0 and α are two parameters of the theory.

Solution of the Problem in the Laplace Transform Domain

By using a non-dimension variables

$$x^* = c_1\eta\, x \quad , \quad \mathrm{u}^* = c_1\eta u \quad , \quad \sigma^*_{\mathrm{ij}} = \frac{\sigma_{ij}}{\mu} \quad , \quad \theta = \frac{\gamma(\mathrm{T}-\mathrm{T}_0)}{(\lambda+2\mu)},$$

$$t^* = c_1^2 \eta t \quad , \quad \tau_0 = c_1^{2\alpha} \eta^\alpha \tau_0 \quad , \quad y^* = c_1 \eta y \quad , \quad v^* = c_1 \eta v,$$

$$\eta = \frac{\rho c_E}{k} \quad , \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}.$$

The equations (2) - (6) in non-dimensional form become (dropping the asterisks for convenience)

$$\sigma_{xx} = \beta^2 e - 2\frac{\partial v}{\partial y} - \beta^2 \theta, \tag{7a}$$

$$\sigma_{yy} = \beta^2 e - 2\frac{\partial u}{\partial x} - \beta^2 \theta, \tag{7b}$$

$$\sigma_{zz} = (\beta^2 - 2)e - \beta^2 \theta, \tag{7c}$$

$$\sigma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{7d}$$

$$\nabla^2 \mathbf{u} + (\beta^2 - 1)\,\text{grad div}\,\mathbf{u} - \beta^2\,\text{grad}\,\theta = \beta^2\,\ddot{\mathbf{u}} \ . \tag{8}$$

$$(\beta^2 - 1)\frac{\partial e}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \beta^2 \frac{\partial \theta}{\partial x} = \beta^2 \frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$(\beta^2 - 1)\frac{\partial e}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \beta^2 \frac{\partial \theta}{\partial y} = \beta^2 \frac{\partial^2 v}{\partial t^2} \tag{10}$$

$$\nabla^2 \theta = \frac{\partial}{\partial t}\left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right)(\theta + \varepsilon e). \tag{11}$$

where $\beta^2 = \dfrac{(\lambda + 2\mu)}{\mu} \quad , \quad \varepsilon_1 = \dfrac{T_0 \gamma^2}{c_E \rho^2 c_0^2}$

using Laplace transform with respect to time with parameter s, to both sides of equations (7)-(11), we obtain

$$\bar{\sigma}_{xx} = \beta^2 \bar{e} - 2\frac{\partial \bar{v}}{\partial y} - \beta^2 \bar{\theta},$$

(12a)

$$\bar{\sigma}_{yy} = \beta^2 \bar{e} - 2\frac{\partial \bar{u}}{\partial x} - \beta^2 \bar{\theta},$$

(12b)

$$\bar{\sigma}_{zz} = (\beta^2 - 2)\bar{e} - \beta^2 \bar{\theta},$$

(12c)

$$\bar{\sigma}_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}$$

(12d)

$$\nabla^2 \bar{\mathbf{u}} + (\beta^2 - 1)\,\text{grad div}\,\bar{\mathbf{u}} - \beta^2\,\text{grad}\,\bar{\theta} = \beta^2 s^2 \bar{\mathbf{u}} \ . \tag{13}$$

$$(\beta^2 - 1)\frac{\partial \bar{e}}{\partial x} + \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) - \beta^2 \frac{\partial \bar{\theta}}{\partial x} = \beta^2 s^2 \bar{u} \tag{14}$$

$$(\beta^2-1)\frac{\partial \bar{e}}{\partial y}+\left(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}\right)-\beta^2\frac{\partial \bar{\theta}}{\partial y}=\beta^2 s^2 \bar{v} \tag{15}$$

$$\nabla^2\bar{\theta}=s\left(1+\tau_0 s^\alpha\right)\left(\bar{\theta}+\varepsilon\,\bar{e}\right). \tag{16}$$

Taking the divergence for both side of equation (13), we obtain

$$\nabla^2\bar{e}-\nabla^2\bar{\theta}=s^2\bar{e}. \tag{17}$$

Eliminating $\bar{\theta}$ between equations (16) and (17), we get

$$\left\{\nabla^4-\left[s^2+s\left(1+\tau_0 s^\alpha\right)(1+\varepsilon)\right]\nabla^2+s^3\left(1+\tau_0 s^\alpha\right)\right\}\bar{e}=0 \tag{18}$$

In a similar manner we can show that $\bar{\theta}$ satisfy the equations

$$\left\{\nabla^4-\left[s^2+s\left(1+\tau_0 s^\alpha\right)(1+\varepsilon)\right]\nabla^2+s^3\left(1+\tau_0 s^\alpha\right)\right\}\bar{\theta}=0 \tag{19}$$

Equation (18) can be factorized as

$$\left(\nabla^2-k_1^2\right)\left(\nabla^2-k_2^2\right)\bar{e}=0 \tag{20}$$

where $k_1^2 \, and \, k_2^2$ are the roots of the characteristic equation:

$$k^4-\left[s^2+s\left(1+\tau_0 s^\alpha\right)(1+\varepsilon)\right]k^2+s^3\left(1+\tau_0 s^\alpha\right)=0$$

In order to solve equation (20), we shall use the exponential Fourier transform with respect to the variable x (denoted by an asterisk) and defined by the relation

$$f^*(q,y,s)=F\,[f(x,y,s)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iqx}\,f(x,y,s)\,dx$$

Applying the exponential Fourier transform to both sides of equation (20), we get

$$\left(D^2-m_1^2\right)\left(D^2-m_2^2\right)\bar{e}=0 \tag{21}$$

where $D=\dfrac{d}{dy}$, $m_1^2=k_1^2+q^2$, $m_2^2=k_2^2+q^2$

The solution of equation (21) is taken as

$$\bar{e}^*=Ak_1^2\,e^{-m_1 y}+Bk_2^2\,e^{-m_2 y} \tag{22}$$

where A and B are some parameters depending on s and q.

Taking the Fourier Transform of both sides of equation (17) and substituting from equation (22), we obtain

$$\bar{\theta}^*=A\left(k_1^2-s^2\right)e^{-m_1 y}+B\left(k_2^2-s^2\right)e^{-m_2 y} \tag{23}$$

In order to obtain the displacement component u, we apply the exponential Fourier transform to both sides of equation (14) to get

$$\left(D^2-n^2\right)\bar{u}^*=iq\left[\left(\beta^2-1\right)\bar{e}^*-\beta^2\bar{\theta}^*\right] \tag{24}$$

where $n^2=\beta^2 s^2+q^2$

Substituting from equations (22) and (23) into equation (24), we obtain

$$\left(D^2-n^2\right)\bar{u}^*=iq\left[\left(k_1^2-\beta^2 s^2\right)Ae^{-m_1 y}+\left(k_2^2-\beta^2 s^2\right)Be^{-m_2 y}\right] \tag{25}$$

The solution of equation (25) is given by

$$\bar{u}^*=Ce^{-ny}-iq\left[A\,e^{-m_1 y}+Be^{-m_2 y}\right] \tag{26}$$

Taking the Laplace and exponential Fourier transforms of equation (1), and useing equations (22), (26), we get

$$\bar{v}^{*} = \frac{-iq}{n} C e^{-ny} - A m_1 e^{-m_1 y} - B m_2 e^{-m_2 y} \tag{27}$$

Substituting from equations (22), (23), (26) and (27) into equations (12), we obtain the transformed stress components in the form

$$\bar{\sigma}^{*}_{xx} = -2iq C e^{-ny} + A\left(\beta^2 s^2 - 2m_1^2\right)e^{-m_1 y} + B\left(\beta^2 s^2 - 2m_2^2\right)e^{-m_2 y} \tag{28}$$

$$\bar{\sigma}^{*}_{yy} = 2iq C e^{-ny} + A\left(n^2 + q^2\right) e^{-m_1 y} + B\left(n^2 + q^2\right) e^{-m_2 y} \tag{29}$$

$$\bar{\sigma}^{*}_{xy} = -\frac{\left(n^2 + q^2\right)}{n} C e^{-ny} + 2iq A m_1 e^{-m_1 y} + 2iq B m_2 e^{-m_2 y} \tag{30}$$

The boundary condition of the problem are taken as

Mechanical boundary condition

$$u(x, y, t) = v(x, y, t) = 0 \quad on \ y = 0 \tag{31}$$

The thermal boundary condition

$$\theta = F(x) H(t) \ \ 0n \ \ y = 0 \tag{32}$$

where $H(t)$ is the Heaviside unit step function and F is a given function of x only

Now applying the boundary condition, we arrive at

$$C - iq A - iq B = 0 \tag{33}$$

$$\frac{-iq}{n} C - A m_1 - B m_2 = 0 \tag{34}$$

$$A\left(k_1^2 - s^2\right) + B\left(k_2^2 - s^2\right) = \frac{F^{*}(q)}{s} \tag{35}$$

By solving equations (33)-(35), we find the expressions for A, B and C. The solution of these equations can be written as

$$A = \frac{\left[m_2 + \frac{q^2}{n}\right] F^{*}(q)}{s\,\Delta}, \quad B = \frac{-\left[m_1 + \frac{q^2}{n}\right] F^{*}(q)}{s\,\Delta}, \quad C = \frac{iq\,[m_2 - m_1] F^{*}(q)}{s\,\Delta}$$

Where, $\Delta = \left(k_1^2 - s^2\right)\left(m_2 + \frac{q^2}{n}\right) - \left(k_2^2 - s^2\right)\left(m_1 + \frac{q^2}{n}\right)$

This completes the solution in the transform domain [4].

Numerical Results

The function $F(x)$ representing the thermal shock was taken as $F(x) = xH(a - |x|)$, which gives.

$$F^{*}(q) = i\sqrt{\frac{2}{\pi}}\left[\frac{a\cos(qa)}{q} - \frac{a\sin(qa)}{q}\right]$$

We shall apply our results to the copper material. The material properties are

$$\lambda = 7.76 \times 10^{10}, \mu = 3.86 \times 10^{10}, \rho = 8954, k = 386, C_E = 381, T_0 = 293,$$

$$\alpha_t = 1.78 \times 10^{-5}, a = 0.5, \tau_0 = 0.01$$

All field quantities temperature, displacement and stress depend on x, y and t. The problem was solved for different values of time $t = \{0.1, 0.2, 0.3\}$ with fixed value of fractional parameter $\alpha = 1$, which are represented by figures (1-5). Dotted lines represent the solution for t = 0.3, double Dotted lines represent the solution for t = 0.2 and solid lines represent the solution when t = 0.1. And different values of the fractional parameter $\alpha = \{0.1, 0.5, 0.99, 1\}$ for fixed value of time $t = 0.1$, which are represented by figures (6-10). Black lines represent the solution for α = 0.1, red lines represent the solution for α = 0.5, yellow lines represent the solution for α = 0.99 and blue lines represent the solution when α = 1.

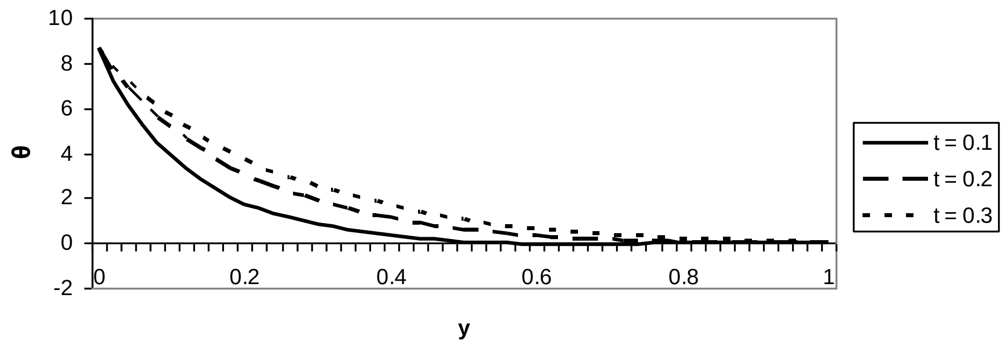

Fig. 1 Temperature Distribution

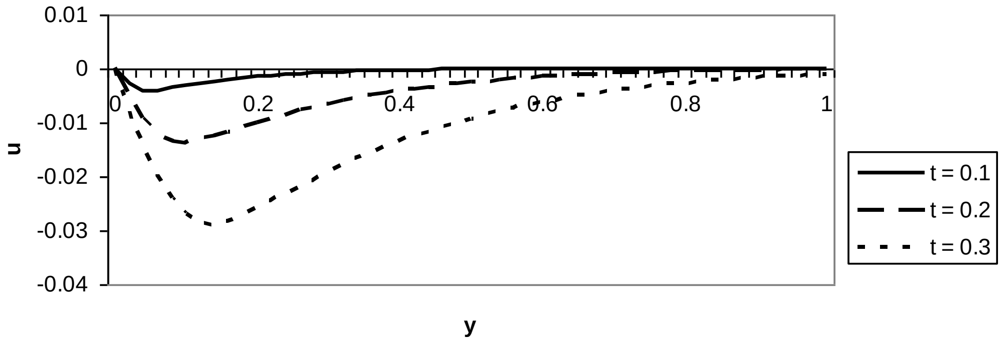

Fig. 2 Displacement Distribution

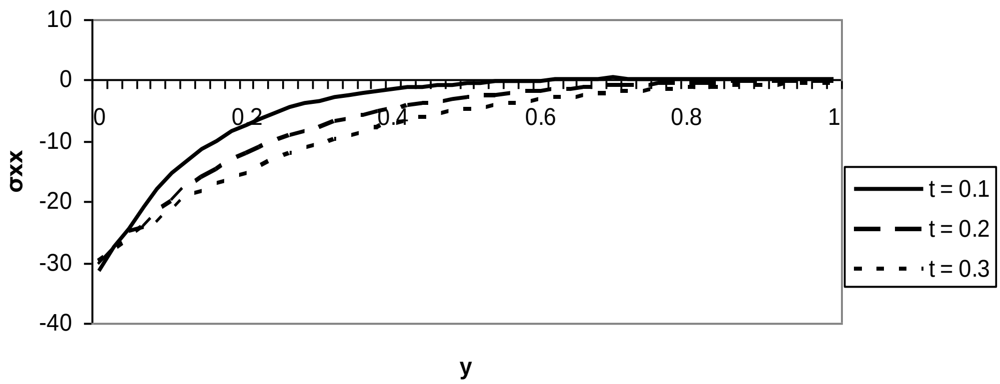

Fig. 3 Shear Stresses Distribution

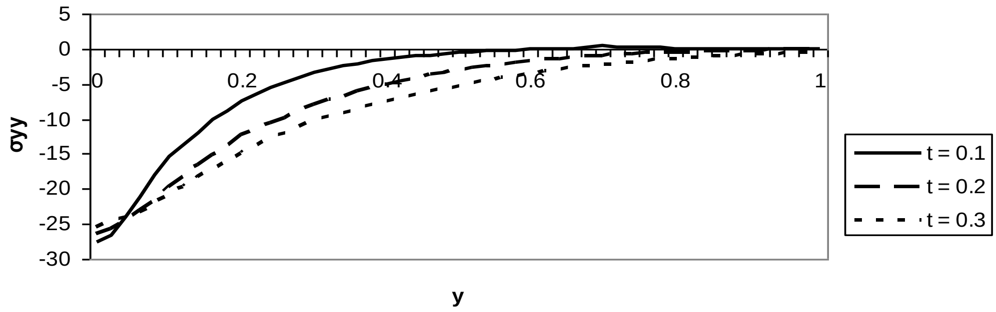

Fig. 4 Normal Stresses Distribution

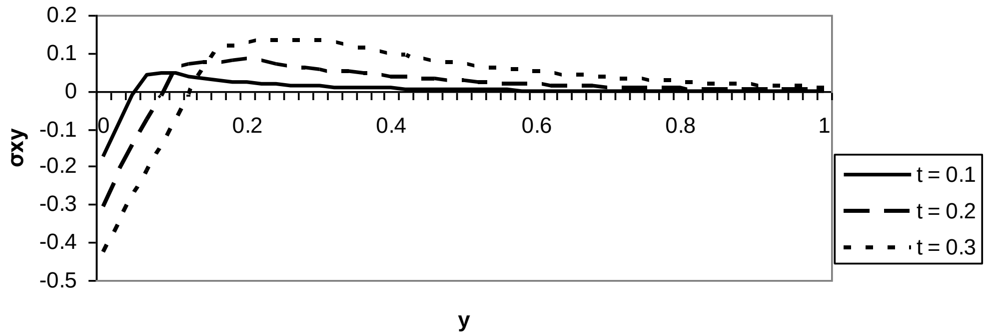

Fig. 5 Stresses Distribution

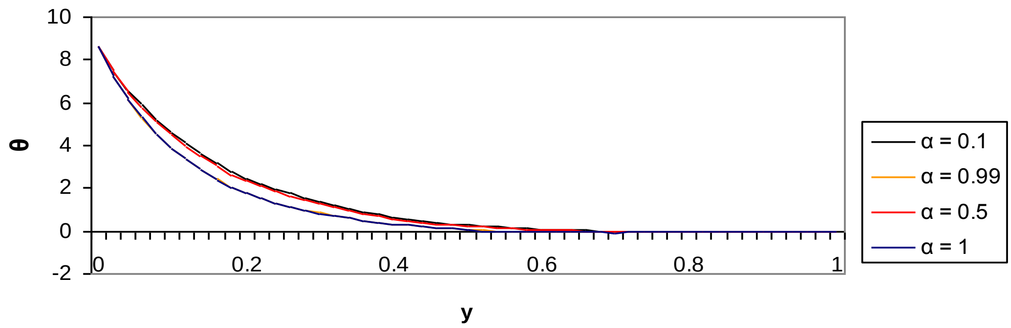

Fig. 6 Temperature Distribution

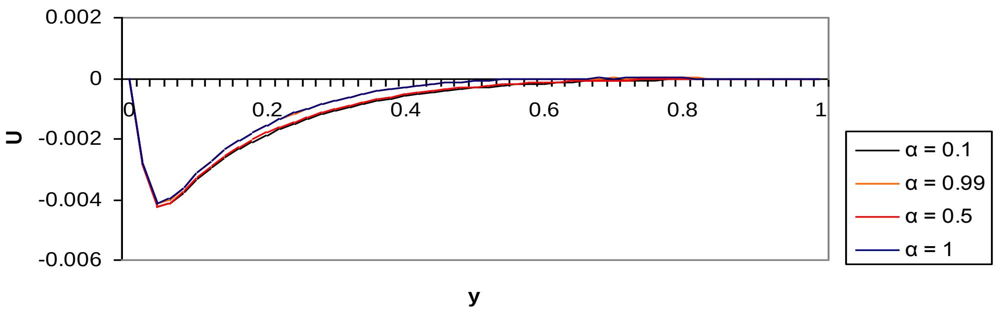

Fig. 7 Desplacement Distribution

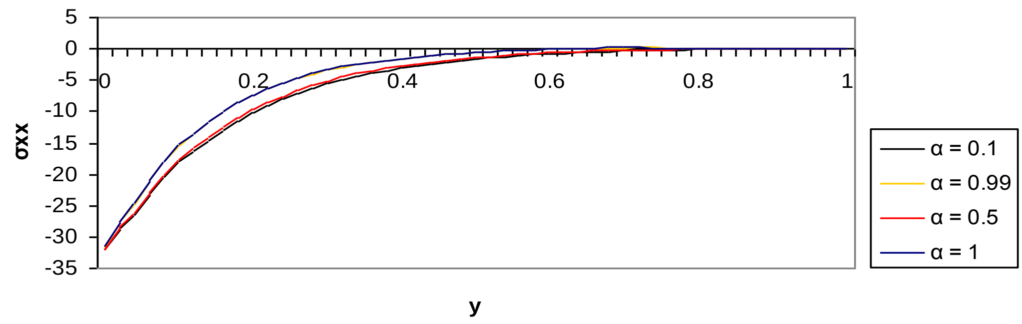

Fig. 8 Shear Stresses Distribution

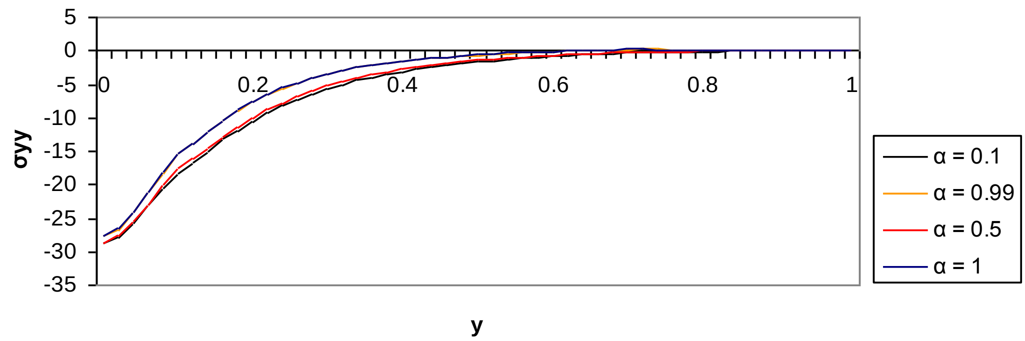

Fig. 9 Normal Stresses Distribution

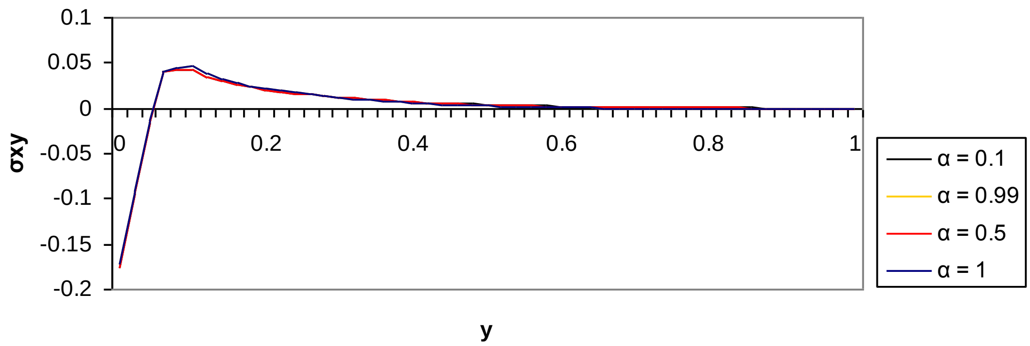

Fig. 10 Stresses Distribution

From figures (1-5), all functions satisfied the boundary conditions. The wave front moves with a finite speed dependent the value of time. This means that all the functions considered have a nonzero value only in a bounded region of space and vanish identically outside this region. All curves decreases regularly until zero reaches the time value, for example in figure 1, when t = 0.1, the curve of temperature reaches at y = 0.43, but when t = 0.2, the curve of temperature reaches at y = 0.65.

From figures (5-10), notice the effect of fraction parameter is appear, the two curves when $\alpha = 0.1 \; or \; 0.5$ are very soon and have the same behavior. Which Leeds to when fractional parameter tends to zero, is given the case of coupled and uncoupled theories of thermo elasticity. I.E the speed of waves are infinite. The two curves when $\alpha = 0.99 \; or \; 1$, are very soon and have the same behavior. Which Leeds to when fractional parameter tends to one is given the case of generalized theory of thermo elasticity. I.E the speed of waves is finite.

Conclusion

A two dimensional fractional order generalized thermoelastic problem for a half-space has been presented. The problem has been solved by using Laplace and Fourier transform techniques. The inversion process is carried out using a numerical method based on Fourier series expansions. The problem takes the effect of fractional parameter. The effect of fractional parameter is appearing.

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