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Research Article

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On the Development of Two-step Implicit Second Derivative Block Methods for the Solution of Initial Value Problems of General Second Order Ordinary Differential Equations

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Abstract In this research work, the development of two-step implicit second derivative block methods for the solution of initial value problems of general second order ordinary differential equations is studied. In the derivation of the method, power series is adopted as basis function to obtain the main continuous scheme through collocation and interpolations approach. Taylor series method was adopted alongside the new method to generate non-overlapping numerical results. The developed method was found to be consistent, convergent and zero-stable. We further computed the order of the newly derived method and plotted the region within which the method is stable. The newly constructed method was applied to solve some systems of second-order stiff ordinary differential equations and from the results obtained, it was clear that the method developed performed better than the existing methods with which we compared our results.

Keywords Collocation, Hybrid, Interpolation, Power Series, Stiff, System

1. Introduction

Most real-life problems that arise in various fields of study be it engineering or sciences are modeled mathematically before they are solved, [1]. These models often lead to differential equations. A differential equation can simply be defined as an equation that contains derivative(s). In other words, it's a relationship involving an independent variable x, a dependent variable y and one or more differential co-efficient of y with respect to x.

Numerous problems such as chemical kinetics, orbital dynamics, circuit and control theory are modeled into second-order differential equations, [2]. It is important to state that mathematical modeling is the art of translating problem from an application area into tractable mathematical formulations whose theoretical and numerical analysis provides insight, answers and guidance, useful for the originating application, [3]. This type of problem can be formulated either in terms of first-order or higher order ODEs. In this article, the system of second-order ODEs of the form;

$$\left.\begin{array}{lll} {}^1y''\,{}^1f\left(x^1, y^1, {}^2y'\right), & {}^1y(x_0) = a_0, & {}^1y'(x_0) = b_0 \\ {}^2y''\,{}^2f\left(x^2, y^2, {}^2y'\right), & {}^2y(x_0) = a_1, & {}^2y'(x_0) = b_1 \\ & \vdots & \\ {}^my''\,{}^mf\left(x^m, y^m, {}^my'\right), & {}^my(x_0) = a_0, & {}^my'(x_0) = b_m \end{array}\right\} \tag{1}$$

shall be considered. The method of solving higher-order ODEs by reducing them to a system of first-order approach involves more functions to evaluate and this leads to computational burden as mentioned in [4-5]. The multistep methods for solving higher-order ODEs directly have been developed by many scholars such as [6-10]. The aim of this paper is to develop a new numerical method for solving systems of second-order stiff ODEs of the form (1).

This paper is organized as follows: in the coming section, we carried out the derivation of the method, where we considered two off-step points through the approach of interpolation and collocation. The details of the analysis of the method were discussed in Section three. In fourth section, some numerical problems were solved and the performance of the developed method was compared with those of the existing methods [13]. Finally, the conclusion was drawn in section five.

2. Derivation of the Method

In this section, a two-step hybrid block method with two off-step points, $x_{n+\frac{3}{4}}$ *and* $x_{n+\frac{7}{4}}$ for solving problems of the form (1) is derived. Let the power series of the form,

$$^{j}y(x)=\sum_{i=0}^{v+m-1}a_i\left(\frac{x-x_n}{h}\right)^i, \quad j=1,\ldots,m. \tag{2}$$

be the approximate solution to equation (1) for $x\in[x_n,x_2]$, where $n=0,1,2,\cdots N-1,$ $a's$ are the real coefficients to be determined, v is the number of collocation points, m is the number of interpolation points and $h=x_n-x_{n-1}$ is a constant step size of the partition of interval $[a,b]$, which is given by $a=x_0<x_1<\cdots<x_N=b$.

Differentiating equation (2) once and twice yield,

$$^{j}y'(x)={}^{j}f\left(x^j,y^j,{}^{j}y\right)=\sum_{i=1}^{v+m-1}\frac{ia_i}{h}\left(\frac{x-x_n}{h}\right)^{i-1}, \quad j=1,\cdots,m. \tag{3}$$

$$^{j}y''(x)={}^{j}f\left(x^j,y^j,{}^{j}y'\right)=\sum_{i=2}^{v+m-1}\frac{i(i-1)a_i}{h^2}\left(\frac{x-x_n}{h}\right)^{i-2}, \quad j=1,\cdots,m. \tag{4}$$

Interpolating equation (2) at the selected intervals, i.e., $x_n, x_{n+\frac{3}{4}}, x_{n+1}, x_{n+\frac{7}{4}}$ *and* x_{n+2} and collocating Equation (4) at all points in the selected interval, i.e., $x_n, x_{n+\frac{3}{4}}, x_{n+1}, x_{n+\frac{7}{4}}$ *and* x_{n+2}, gives the following equations which can be written in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & \frac{3}{2h} & \frac{27}{16h} & \frac{27}{16h} & \frac{81}{256h} & \frac{729}{512h} & \frac{5103}{4096h} & \frac{2187}{2048h} & \frac{59049}{65536h} & \frac{98415}{131072h} \\ 0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \frac{4}{h} & \frac{5}{h} & \frac{6}{h} & \frac{7}{h} & \frac{8}{h} & \frac{9}{h} & \frac{10}{h} \\ 0 & \frac{1}{h} & \frac{7}{2h} & \frac{147}{4h} & \frac{343}{16h} & \frac{12005}{256h} & \frac{5044}{512h} & \frac{823543}{4096h} & \frac{823543}{2048h} & \frac{51883209}{65536h} & \frac{201768035}{131072h} \\ 0 & \frac{1}{h} & \frac{4}{h} & \frac{12}{h} & \frac{32}{h} & \frac{80}{h} & \frac{192}{h} & \frac{448}{h} & \frac{1024}{h} & \frac{2304}{h} & \frac{5120}{h} \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{9}{2h^2} & \frac{27}{h^2} & \frac{135}{16h^2} & \frac{1215}{128h^2} & \frac{5103}{512h^2} & \frac{5103}{512h^2} & \frac{15309}{32h^2} & \frac{295245}{32768h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} & \frac{90}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{21}{2h^2} & \frac{147}{4h^2} & \frac{1715}{16h^2} & \frac{36015}{128h^2} & \frac{352947}{512h^2} & \frac{823543}{512h^2} & \frac{7411887}{2048h^2} & \frac{259416045}{32768h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{48}{h^2} & \frac{160}{h^2} & \frac{480}{h^2} & \frac{1344}{h^2} & \frac{3584}{h^2} & \frac{9216}{h^2} & \frac{23040}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{pmatrix} = \begin{pmatrix} {}^j y_n \\ {}^j y_{n+\frac{3}{4}} \\ {}^j y_{n+1} \\ {}^j y_{n+\frac{7}{4}} \\ {}^j y_{n+2} \\ {}^j f_n \\ {}^j f_{n+\frac{3}{4}} \\ {}^j f_{n+1} \\ {}^j f_{n+\frac{7}{4}} \\ {}^j f_{n+2} \end{pmatrix} \quad (5)$$

Applying the Gaussian elimination method on equation (5) gives the coefficients $a_i's, \quad for \quad i = 0(1)10$. These values are then substituted into equation (2) to give the implicit continuous hybrid method of the form:

$$ {}^j y(x) = \sum_{i=\frac{3}{4},\ \frac{7}{4}} {}^j\beta_i(x)^j f_{n+i} + \sum_{i=0}^{2} {}^j\beta_i(x)^j f_{n+i}, \quad j = 1, \cdots, m. \quad (6)$$

Differentiating equation (6) once yields,

$$ {}^j y'(x) = \sum_{i=\frac{3}{4},\ \frac{7}{4}} \frac{d}{dx} {}^j\beta_i(x)^j f_{n+i} + \sum_{i=0}^{2} \frac{d}{dx} {}^j\beta_i(x)^j f_{n+i}, \quad j = 1, \cdots, m. \quad (7)$$

where the continuous schemes are given by;

$$ {}^j\alpha_0 = 0 $$

$$^{j}\beta_0 = x - x_n - \frac{15605}{1764}\frac{(x-x_n)^3}{h^2} + \frac{3808013}{148176}\frac{(x-x_n)^4}{h^3} - \frac{6947149}{185220}\frac{(x-x_n)^5}{h^4} + \frac{1047937}{31752}\frac{(x-x_n)^6}{h^5} - \frac{1182388}{64827}\frac{(x-x_n)^7}{h^6} + \frac{57497}{9261}\frac{(x-x_n)^8}{h^7} - \frac{99328}{83349}\frac{(x-x_n)^9}{h^8} + \frac{4576}{46305}\frac{(x-x_n)^{10}}{h^9}$$

$$^{j}\beta_{\frac{3}{4}} = -\frac{476672}{1125}\frac{(x-x_n)^3}{h^2} + \frac{1224832}{675}\frac{(x-x_n)^4}{h^3} - \frac{57907072}{16875}\frac{(x-x_n)^5}{h^4} + \frac{36835712}{10125}\frac{(x-x_n)^6}{h^5} - \frac{1560704}{675}\frac{(x-x_n)^7}{h^6} + \frac{2958656}{3375}\frac{(x-x_n)^8}{h^7} - \frac{5564416}{30375}\frac{(x-x_n)^9}{h^8} + \frac{274432}{16875}\frac{(x-x_n)^{10}}{h^9}$$

$$^{j}\beta_1 = \frac{3724}{9}\frac{(x-x_n)^3}{h^2} - \frac{15785}{9}\frac{(x-x_n)^4}{h^3} + \frac{88895}{27}\frac{(x-x_n)^5}{h^4} - \frac{280928}{81}\frac{(x-x_n)^6}{h^5} + \frac{414112}{189}\frac{(x-x_n)^7}{h^6} - \frac{22304}{27}\frac{(x-x_n)^8}{h^7} + \frac{41728}{243}\frac{(x-x_n)^9}{h^8} - \frac{2048}{135}\frac{(x-x_n)^{10}}{h^9}$$

$$^{j}\beta_{\frac{7}{4}} = -\frac{8704}{441}\frac{(x-x_n)^3}{h^2} + \frac{294272}{3087}\frac{(x-x_n)^4}{h^3} - \frac{9546112}{46305}\frac{(x-x_n)^5}{h^4} + \frac{1001600}{3969}\frac{(x-x_n)^6}{h^5} - \frac{12069760}{64827}\frac{(x-x_n)^7}{h^6} + \frac{759872}{9261}\frac{(x-x_n)^8}{h^7} - \frac{1656832}{83349}\frac{(x-x_n)^9}{h^8} + \frac{94208}{46305}\frac{(x-x_n)^{10}}{h^9}$$

$$^{j}\beta_2 = \frac{19257}{500}\frac{(x-x_n)^3}{h^2} - \frac{72681}{400}\frac{(x-x_n)^4}{h^3} + \frac{956969}{2500}\frac{(x-x_n)^5}{h^4} - \frac{1365623}{3000}\frac{(x-x_n)^6}{h^5} + \frac{56964}{175}\frac{(x-x_n)^7}{h^6} - \frac{17353}{125}\frac{(x-x_n)^8}{h^7} + \frac{36608}{1125}\frac{(x-x_n)^9}{h^8} - \frac{2016}{625}\frac{(x-x_n)^{10}}{h^9}$$

$$^{j}\gamma_0 = \frac{1}{2}(x-x_n)^2 - \frac{143}{63}\frac{(x-x_n)^3}{h} + \frac{34981}{7056}\frac{(x-x_n)^4}{h^2} - \frac{5687}{882}\frac{(x-x_n)^5}{h^3} + \frac{8063}{1512}\frac{(x-x_n)^6}{h^4} - \frac{8756}{3087}\frac{(x-x_n)^7}{h^5} + \frac{415}{441}\frac{(x-x_n)^8}{h^6} - \frac{704}{3969}\frac{(x-x_n)^9}{h^7} + \frac{32}{2205}\frac{(x-x_n)^{10}}{h^8}$$

$$^{j}\gamma_{\frac{3}{4}} = -\frac{12544}{225}\frac{(x-x_n)^3}{h} + \frac{10304}{45}\frac{(x-x_n)^4}{h^2} - \frac{471488}{1125}\frac{(x-x_n)^5}{h^3} + \frac{292288}{675}\frac{(x-x_n)^6}{h^4} - \frac{84928}{315}\frac{(x-x_n)^7}{h^5} + \frac{22624}{225}\frac{(x-x_n)^8}{h^6} - \frac{41984}{2025}\frac{(x-x_n)^9}{h^7} + \frac{2048}{1125}\frac{(x-x_n)^{10}}{h^8}$$

$$ {}^{j}\gamma_{1} = -\frac{196}{3}\frac{(x-x_n)^3}{h} + \frac{854}{3}\frac{(x-x_n)^4}{h^2} - \frac{24733}{45}\frac{(x-x_n)^5}{h^3} + \frac{32137}{54}\frac{(x-x_n)^6}{h^4} - \frac{3472}{9}\frac{(x-x_n)^7}{h^5} + \frac{1340}{9}\frac{(x-x_n)^8}{h^6} - \frac{2560}{81}\frac{(x-x_n)^9}{h^7} + \frac{128}{45}\frac{(x-x_n)^{10}}{h^8} $$

$$ {}^{j}\gamma_{\frac{7}{4}} = -\frac{256}{21}\frac{(x-x_n)^3}{h} + \frac{8384}{147}\frac{(x-x_n)^4}{h^2} - \frac{262336}{2205}\frac{(x-x_n)^5}{h^3} + \frac{3776}{27}\frac{(x-x_n)^6}{h^4} - \frac{304576}{3087}\frac{(x-x_n)^7}{h^5} + \frac{18272}{441}\frac{(x-x_n)^8}{h^6} - \frac{37888}{3969}\frac{(x-x_n)^9}{h^7} + \frac{2048}{2205}\frac{(x-x_n)^{10}}{h^8} $$

$$ {}^{j}\gamma_{2} = -\frac{147}{50}\frac{(x-x_n)^3}{h} + \frac{1113}{80}\frac{(x-x_n)^4}{h^2} - \frac{3677}{125}\frac{(x-x_n)^5}{h^3} + \frac{7027}{200}\frac{(x-x_n)^6}{h^4} - \frac{884}{35}\frac{(x-x_n)^7}{h^5} + \frac{271}{25}\frac{(x-x_n)^8}{h^6} - \frac{64}{25}\frac{(x-x_n)^9}{h^7} + \frac{32}{125}\frac{(x-x_n)^{10}}{h^8} $$

3. Analysis of the Method

In this section, the basic properties of the method derived hall be analyzed.

3.1. Order and error Constants of the Method

According to [11], the order of the new method in equation (6) is obtained by using the Taylor series and it is found that the developed method is of uniform order ten, with an error constants vector given by,

$$ C_{10} = \left[5.4779\times10^{-9} \quad 5.4845\times10^{-8} \quad 6.0578\times10^{-9} \quad 6.0891\times10^{-9}\right]^T $$

3.2. Consistency

Definition 3.1[2]: The hybrid block method (6) is said to be consistent if it has an order more than or equal to one, i.e., $P \geq 1$. Therefore, the method is consistent since it is of order ten.

3.3. Zero Stability

Definition 3.2[2]: The hybrid block method (6)said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1 \;\; and \;\; if \;\; |r_z| = 1$, then the multiplicity of r_z must not be greater than two.

In order to find the zero-stability of hybrid block method (6), we only consider the first characteristic polynomial of the method according to Definition 3.2 as follows,

$$ \Pi(r) = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| = r^3(r-1) $$

which implies $r = 0, 0, 0, 1$. Hence, the method is zero-stable since $|r_z| \leq 1$.

3.4. Convergence

Theorem 3.1 [12]: Consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method (6) is consistent and also zero stable, it implies the method is convergent for all points.

3.5. Regions of Absolute Stability

The absolute stability region of the new method is shown in the figure below.

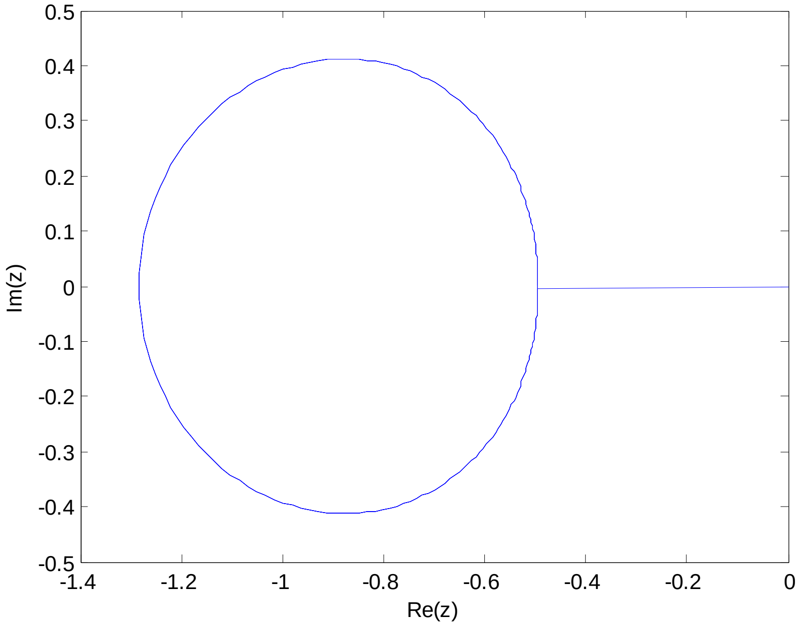

Figure 3.1: Region of Absolute Stability

4. Numerical Implementation

In this section, the efficiency and the performance of the general two-step implicit hybrid block method with order $p=10$ is investigated on two test problems. We present some numerical examples solved by [13]. The performance of the new method is examined using the following two systems of second-order initial value problems of ordinary differential equations. Tables 4.1and 4.2 show the comparison of the results obtained by the new methods with that of the existing method [13]in terms of absolute and relative errors.

Problem 4.1

Consider the stiff system

$$\left.\begin{array}{ll} y_1' = 998y_1 + 1998y_2 & y_1(0)=1 \\ y_2' = -999y_1 - 1999y_2 & y_2(0)=0, \quad h=0.1 \end{array}\right\} \tag{8}$$

with Exact Solution

$$\left.\begin{array}{l} y_1(x) = 2e^{-x} - e^{-1000x} \\ y_2(x) = -e^{-x} - e^{-1000x} \end{array}\right\} \tag{9}$$

Source: [13]

Table 4.1: Comparison of results of the proposed method with that of [13]

x	**Errors in [13],** $K=2$				**Errors in New method,** $K=2$			
	Absolute errors		**Relative errors**		**Absolute errors**		**Relative errors**	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	$2.43E^{-2}$	$2.43E^{-2}$	$1.34E^{-2}$	$2.69E^{-2}$	$3.00E^{-1}$	$3.47E^{-3}$	$1.95E^{-1}$	$3.83E^{-3}$
0.2	$3.87E^{-2}$	$3.81E^{-2}$	$2.14E^{-2}$	$4.73E^{-2}$	$3.17E^{-2}$	$3.17E^{-2}$	$1.94E^{-2}$	$3.87E^{-2}$
0.3	$9.31E^{-4}$	$9.85E^{-4}$	$6.28E^{-4}$	$1.26E^{-3}$	$1.67E^{-4}$	$1.13E^{-4}$	$7.88E^{-5}$	$1.53E^{-4}$
0.4	$1.51E^{-5}$	$1.51E^{-3}$	$1.13E^{-3}$	$2.25E^{-3}$	$1.01E^{-3}$	$1.01E^{-5}$	$7.53E^{-4}$	$1.50E^{-3}$

0.5	$2.32E^{-5}$	$2.20E^{-5}$	$1.91E^{-5}$	$3.63E^{-5}$	$1.17E^{-5}$	$7.63E^{-6}$	$9.67E^{-6}$	$1.26E^{-5}$
0.6	$6.99E^{-5}$	$7.14E^{-5}$	$6.37E^{-5}$	$1.30E^{-4}$	$3.91E^{-5}$	$3.56E^{-5}$	$3.56E^{-5}$	$6.48E^{-5}$
0.7	$2.15E^{-5}$	$1.22E^{-5}$	$2.16E^{-5}$	$2.46E^{-5}$	$1.94E^{-4}$	$4.58E^{-6}$	$1.95E^{-4}$	$9.22E^{-6}$
0.8	$2.34E^{-5}$	$1.46E^{-5}$	$2.60E^{-5}$	$3.24E^{-5}$	$8.92E^{-6}$	$4.73E^{-2}$	$9.93E^{-6}$	$1.05E^{-1}$
0.9	$2.17E^{-5}$	$160E^{-5}$	$2.67E^{-5}$	$3.95E^{-5}$	$2.04E^{-6}$	$4.54E^{-6}$	$2.52E^{-6}$	$1.12E^{-5}$
1.0	$1.97E^{-5}$	$1.48E^{-5}$	$2.68E^{-5}$	$4.03E^{-5}$	$1.02E^{-5}$	$4.16E^{-6}$	$1.39E^{-5}$	$1.13E^{-5}$

Problem 4.2

Consider the stiff system

$$\left.\begin{array}{ll} y_1' = 198y_1 + 199y_2 & y_1(0) = 1 \\ y_2' = -398y_1 - 399y_2 & y_2(0) = -1, \quad h = 0.1 \end{array}\right\} \qquad (10)$$

with Exact Solution

$$\left.\begin{array}{l} y_1(x) = e^{-x} \\ y_2(x) = -e^{-x} \end{array}\right\} \qquad (11)$$

Source: [13]

Table 4.2: Comparison of results of the proposed method with that of [13]

x	**Errors in [13],** $K=2$				**Errors in New method,** $K=2$			
	Absolute errors		**Relative errors**		**Absolute errors**		**Relative errors**	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	$3.61E^{-7}$	$3.60E^{-7}$	$3.99E^{-7}$	$3.98E^{-7}$	$8.63E^{-8}$	$1.07E^{-7}$	$9.54E^{-8}$	$1.18E^{-7}$
0.2	$3.21E^{-7}$	$3.30E^{-7}$	$3.55E^{-7}$	$3.91E^{-7}$	$3.46E^{-8}$	$7.18E^{-8}$	$4.23E^{-8}$	$8.77E^{-8}$
0.3	$6.28E^{-7}$	$3.27E^{-7}$	$8.47E^{-7}$	$8.47E^{-7}$	$7.32E^{-8}$	$9.02E^{-8}$	$9.88E^{-8}$	$1.22E^{-7}$
0.4	$5.65E^{-7}$	$5.65E^{-7}$	$7.63E^{-7}$	$8.43E^{-7}$	$2.45E^{-8}$	$5.51E^{-8}$	$3.65E^{-8}$	$8.22E^{-8}$
0.5	$6.69E^{-7}$	$6.68E^{-7}$	$1.10E^{-6}$	$1.10E^{-6}$	$6.35E^{-8}$	$7.74E^{-8}$	$1.05E^{-7}$	$1.28E^{-7}$
0.6	$6.03E^{-7}$	$6.02E^{-7}$	$1.10E^{-6}$	$1.10E^{-6}$	$1.70E^{-8}$	$4.20E^{-8}$	$3.10E^{-8}$	$7.65E^{-8}$
0.7	$5.92E^{-7}$	$5.92E^{-7}$	$1.19E^{-6}$	$1.20E^{-6}$	$5.46E^{-8}$	$6.59E^{-8}$	$1.10E^{-7}$	$1.33E^{-7}$
0.8	$5.36E^{-7}$	$5.37E^{-7}$	$1.10E^{-6}$	$1.19E^{-6}$	$1.17E^{-8}$	$3.22E^{-8}$	$2.61E^{-8}$	$1.17E^{-8}$
0.9	$7.38E^{-7}$	$7.38E^{-7}$	$1.82E^{-6}$	$1.81E^{-6}$	$4.71E^{-8}$	$5.64E^{-8}$	$1.16E^{-7}$	$1.39E^{-7}$

5. Conclusion

The development of two-step implicit second derivative for the solution general second order ordinary differential equations has been derived via the interpolation and collocation approach. The absolute and relative errors arising from Problems 4.1 and 4.2 using the new method were compared with the existing method [13]. It is evident from the results obtained that the newly derived method performs better than the existing method [13]. The method is also desirable by virtue of possessing high order of accuracy. The developed method is also consistent, zero-stable and convergent.

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