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Mathematics of Bifurcation Applied to Dynamical Systems in the Physical Sciences

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Abstract A stable fixed point in a dynamical system is likely to persist in that state, even in the face of minor disturbances or perturbations. It is observed that varying the parameters in a dynamical system evolution function can result in a radical change in systems behavior called a bifurcation [1]. In the case of population dynamics bifurcation is a transition from a fixed population to an oscillation between high and low populations. In this article we will focus on the mathematics of bifurcation in the science of dynamical systems behavior. In particular we discuss the bifurcation behavior of population dynamical system in biology, weather prediction model and chemical reaction system.

Keywords Bifurcation, dynamical system, perturbation, evolution function, state of a system

1. Introduction

To explain, analyze and understand dynamical systems behavior in both natural and artificial systems we use the mathematics of chaos theory [2] and bifurcation. Chaos theory is a scientific discipline which is focused on the study of nonlinear systems, which are generally complex and unpredictable. Chaos theory deals with dynamical systems that evolve in time and characterized by properties such as sensitivity to initial conditions and topological mixing. The cause of unpredictability in nonlinear systems is extreme sensitivity to initial conditions-what is referred to as the butterfly effect. The concept means that with a complex non-linear system, very small changes in the starting conditions of a system will result in dramatically different and large changes in the outputs for that system. Chaos theory describes complex motion and the dynamics of sensitive systems. Chaotic systems are mathematically deterministic but nearly impossible to predict. Chaos is more evident in long-term systems than in short-term systems. Behavior in chaotic systems is a periodic, meaning that no variable describing the state of the system undergoes a regular repetition of values. A chaotic system can actually evolve in a way that appears to be smooth and ordered, however Chaos refers to the issue of whether or not it is possible to make accurate long-term predictions of any system if the initial conditions are known to an accurate degree [3]. As a result of sensitivity to initial conditions like initial position and velocity the French mathematician Henri Poincare concluded that he could not predict the trajectories of planets in the solar system including the earth [4]. The uncertainty in the movement of the earth in Poincare's model contributes to unpredictability in climate change. In the same way the American meteorologist Edward Lorenz discovered that a simple model of heat convection possesses intrinsic unpredictability [5]. Chaotic models are generally extremely sensitive to initial conditions and characterized by disequilibrium, bifurcation [6-7].

2. Preliminaries and Definition of Terms

(i) BIFURCATION: A bifurcation is sudden and unexpected changes in the behavior of a dynamical system for small changes in parameters.

(ii) MAP: A function whose domain (input space) and range (output space) are the same will be called a **map**. Let x be a point and let f be a map.
(iii) ORBIT OF A MAP: The **orbit** of x under f is the set of points $\{ x, f(x), f^2(x), \ldots\ldots\ldots\ldots \}$.
(iv) FIXED POINT: The starting point x for the orbit is called the initial value of the orbit. A point p is a **fixed point** of the map f if $f(p) = p$.
(v) SINK: let f be a map on $\mathbb{R}$ and let p be a real number such that $f(p) = p$ if all points sufficiently close to p are attracted to p, then p is called a sink or an attracting fixed point. Most precisely, if there is an $\in > 0$ such that for all x in the epsilon neighbourhood $N \in (p)$, $\lim_{K\to\infty} f^k(x) = p$ then p is a **sink** or **attracting point**.
(vi) SOURCE: If all points sufficiently close to p are repelled from p, then p is called a **source or a repelling fixed point**. Most precisely, if there is an epsilon neighborhood $N \in (p)$ such that each x in $N \in (p)$ except for p itself eventually maps outside of $N \in (p)$, then p is a **source**. For more than one dimensional state space, there is a fixed point called a saddle. A **saddle** has at least one attracting direction and at least one repelling direction. A saddle exhibits sensitive dependence on initial conditions, because of the neighboring initial conditions that escape along the repelling direction.
(vii) PERIODIC POINT: let f be a map on $\mathbb{R}$. We call p a **periodic point of period k or period- k point** if $f^k(p) = p$, and if k is the smallest such positive integer. The orbit with initial point p (which consists of k points) is called a **periodic orbit of period k or period- k orbit.**
(viii) SENSITIVE DEPENDENCE: Sensitive dependence on initial conditions in chaos theory means that small variations in the initial conditions of a dynamical system produce large variations in the long term behavior of the system.
(**ix**) **STEADY STATE BIFURCATION:** This is where equilibrium becomes unstable and a new equilibrium appears around it.
(**x**) **HOPF BIFURCATION:** This is where equilibrium becomes unstable and gives rise to a periodic state.
(**xi**) **PITCHFORK BIFURCATION:** This is characterized by the passage from one stable steady state to an unstable steady state and the simultaneous appearance of two stable steady states.

2. Stability of Dynamical Systems

2.1. Stability of Fixed and Periodic Points

A stable fixed point has the property that points near it are moved even closer to the fixed point under the dynamical system. These fixed points are the attracting fixed points. For an unstable fixed point, nearby points move away as time progresses. The fixed point is a source or a saddle. The stability or otherwise of a fixed or periodic point is related to its derivative as stated in the following properties:

Property 1: let f be a continuously differentiable map on $\mathbb{R}$, and assume that p is a fixed point of f : (i) if $|\dot{f}(p)|<1$, then p is a sink (ii) if $|\dot{f}(p)| > 1$, then p is a source.

Property 2: The periodic orbit $\{p_1, p_2, \ldots\ldots\ldots, p_k\}$is a sink if $|\dot{f}(p_k) \ldots\ldots \dot{f}(p_1)| < 1$ *and* source if $|\dot{f}(p_k) \ldots\ldots \dot{f}(p_1)| > 1$.

3. Bifurcations in Dynamical Systems

3.1 Bifurcation in Population fluctuation of the flour beetle Tribolium

The following is a model of population fluctuation in the flour beetle *Tribolium*. The newly hatched larva spends two weeks feeding before entering a pupa stage of about the same length. The beetle exits the pupa stage as an adult. Let the numbers of larvae, pupae, and adults at any given time t be denoted by L_t, P_t, and A_t respectively. After the unit time of two weeks the model for the three beetle populations is given by:

$$L_{t+1} = bA_t \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$P_{t+1} = L_t(1-\mu_l) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$A_{t+1} = P_t(1-\mu_p) + (1-\mu_a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where b is the birth rate of the species (the number of new larvae per adult after each unit time), μ_l, μ_p *and* μ_a are the death rates of the larvae, pupa, and adult, respectively. The above is a three dimensional discrete map. *Tribolium* adds an interesting twist to the above model: cannibalism caused by overpopulation stress. Under conditions of overcrowding, adults, adults will eat pupae and unhatched eggs (future larvae); larvae will also eat eggs. Incorporating these into above model yields:

$$L_{t+1} = bA_t \exp(-c_{ea}A_t - c_{e1}L_t) \ldots \ldots \ldots \ldots (4)$$

$$P_{t+1} = L_t(1-\mu_l) \ldots \ldots \ldots \ldots (5)$$

$$A_{t+1} = P_t(1-\mu_p)\exp(-c_{pa}A_t) + A_t(1-\mu_a) \ldots \ldots \ldots \ldots (6)$$

From population experiments we get the following parameter values:
$c_{el} = 0.012,\ c_{ea} = 0.009,\ c_{pa} = 0.004,\ \mu_l = 0.267,\ \mu_p = 0$ and b = 7.48 with mortality rate of the adult as μ_a = 0.0036. From a graph of adult mortality rate against larvae population we get that for relatively low mortality rates, the larvae population reaches a steady or equilibrium state (fixed point). For $\mu_a > .1$ (representing a death rate of 10% of the adults over each two week period), the model shows oscillation between two widely different states. This is a 'boom' and 'bust' cycle [8]. A low population (bust) leads to uncrowded living conditions and increased growth (boom) at the next generation which eventually lead to overcrowding, cannibalism, catastrophic decline and then a repeat of the cycle. A bifurcation is a change from a fixed population to an oscillation between high and low populations. For the above model the population changes from period-doubling bifurcation (near $\mu_a = 0.1$) to period-halving bifurcation (when $\mu_a \approx 0.6$) and then to chaos (near $\mu_a = 1$) or adult death rate of 100%.

3.2. Hopf Bifurcations from Chemical Reactions

A bifurcation is sudden and unexpected changes in the behavior of a system for small changes in parameters. In an Andronov-Hopf bifurcation a family of periodic orbits bifurcates from a path of equilbria that changes stability at the bifurcation point. Hopf bifurcation is where an equilibrium becomes unstable and give rise to a periodic state. As an example of Hopf bifurcation consider the experiment designed to explore oscillatory phenomena in the dissolving of iron in a sulphuric acid solution. A 99.99% iron rod with a diameter of 3mm is lowered into 200ml of the acid. When a potential difference is applied, the current of the electrochemical system is a measure of the overall reaction rate between the electrode surface and the electrolytic solution. The behavior of the current as a function of time shows considerable complication at certain parameter settings. The electrical potential applied to the electrode is used as a bifurcation parameter in this experiment. The behavior observed shows a short increase in current followed by a constant current which changes to small irregular oscillations which finally leads to clear periodic oscillations as potential is increased. The behavior is chaotic and in this case it is called a Hopf bifurcation [8].

3.3. Bifurcation in Edward Lorenz Weather Prediction System

Edward Lorenz an American meteorologist in 1961 proposed the following model of convection, similar to the swirls of cream in a hot cup of coffee [5]. The models are as specified in equations and 7, 8, 9.

$$\frac{dx}{dt} = -10x + 10y \ldots \ldots \ldots \ldots (7)$$

$$\frac{dy}{dt} = 28x - y + xy \ldots \ldots \ldots \ldots (8)$$

$$\frac{dz}{dt} = -8/3x + xy \ldots \ldots \ldots \ldots (9)$$

The points *x*, *y*, *z* correspond to the position of a point in geometric space at time *t*. Lorenz system of differential equations is unsolvable except by numerical means. It was discovered that the solutions are sensitive to initial conditions. It has been observed that weather is a chaotic dynamical system and hence long term prediction is not possible [9].
In general the Lorenz equation is given by:

$$W(x,y,z) = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}\sigma(y-x)\\ \rho x - y - xz\\ xy - \beta z\end{bmatrix} \ldots \ldots \ldots \ldots (10)$$

where Lorenz fixed $\sigma = 10$, $\beta = \frac{8}{3}$, the "Raleigh number" $\rho = 28$, $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, and $\dot{z} = \frac{dz}{dt}$. For $0 < \rho < 1$ $the\ equilbrium$ $EQ_0 = (0,0,0)$ at the origin is **attractive.** At $\rho = 1$ it undergoes a **pitchfork bifurcation** into a pair of equilbria given by:

$EQ_{1,2} = (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1)$. For $\rho = 24.74, EQ_{1,2}$ undergoes **Hopf bifurcation** [8].

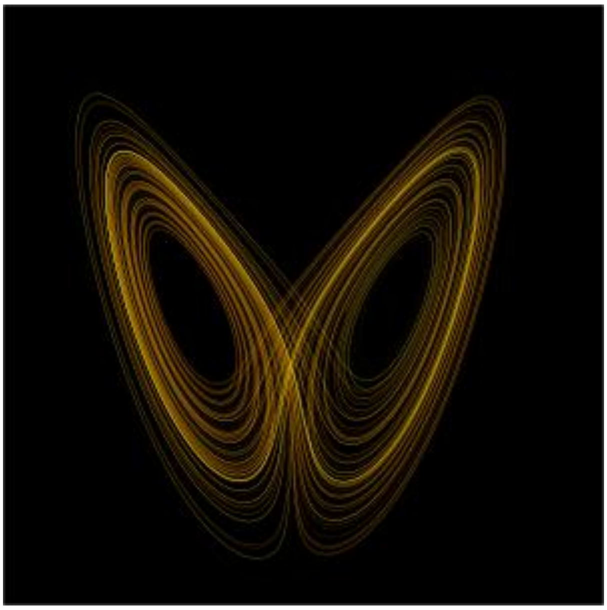
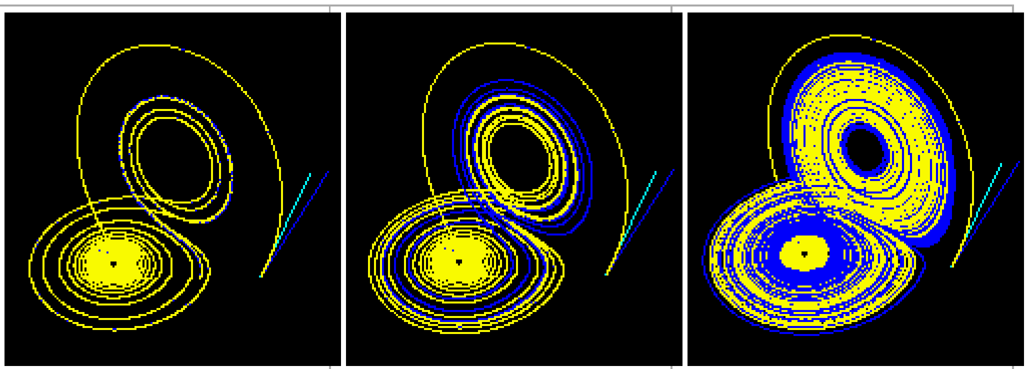


Figure 1: Lorenz strange attractor

The Butterfly diagram in fig1 above is a diagrammatic representation of the Butterfly effect and is referred as the Lorenz attractor. The Butterfly Effect is a phrase that encapsulates the more technical notion of sensitive dependence on initial conditions in chaos theory. The idea is that small variations in the initial conditions of a dynamical system produce large variations in the long term behavior of the system. Edward Lorenz first analyzed the butterfly effect in a 1963 paper on weather simulation and prediction [9]. The practical consequence of the butterfly effect is that complex systems such as the weather are difficult to predict past a certain time range - approximately a week, in the case of weather. This is because any finite model that attempts to simulate a system must necessarily truncate some information about the initial conditions - for example, when simulating the weather, one would not be able to include the wind coming from every butterfly's wings. In all practical cases, defects in the knowledge of the initial conditions and deficiencies in the model are equally important sources of error. In a chaotic system, these errors are magnified as the simulation progresses. Thus the predictions of the simulation are useless after a certain finite amount of time. Lorenz and Poincare models are historically regarded as the foundation of chaos theory. The model revealed the unpredictability of the weather and hence climate change. It is a model of climate that is both stochastic and deterministic. To illustrate the extreme sensitivity of the Lorenz model to initial conditions consider the starting points (1,1,10) and (1,1,10.01). There is a difference of 0.01 between the 3rd component of the two points at t= 0. If we plot the butterfly diagram of these two points from t=0 to t=7.5 we will observe that the two curves fly apart.

The following figure illustrates sensitive dependence on initial conditions.

Sensitive dependence on the initial condition		
Time t=1 (Enlarge)	**Time t=2** (Enlarge)	**Time t=3** (Enlarge)
Fig 2	Fig 3	Fig 4



These figures 2, 3 and 4 — made using ρ=28, σ = 10 and β = 8/3 — show three time segments of the 3-D evolution of 2 trajectories (one in blue or dark, the other in yellow or white) in the Lorenz attractor starting at two initial points that differ only by 10^{-5} in the *x*-coordinate. Initially, the two trajectories seem coincident (only the yellow one can be seen, as it is drawn over the blue one) but, after some time, the divergence is obvious.

In general the Lorenz equation is given by:

$W(x,y,z) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - \beta z \end{bmatrix}$, where Lorenz fixed $\sigma = 10,\ \beta = \frac{8}{3}$, *the* "Raleigh number" $\rho = 28,\ \dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, and $\dot{z} = \frac{dz}{dt}$.

Edward Lorenz in introducing chaos into science and mathematics asked the question: can the flap of a butterfly's wing in Brazil cause a tornado in Texas, U.S.A? The question can be rephrased into: can the flap of a locust's wing in Konduga Borno State Nigeria, cause the water in the lake Chad Basin which covers five countries in west and central Africa to diminish [10].

4. Conclusion

In this paper we presented chaotic dynamical systems and their corresponding behaviors as exhibited by the nature of bifurcation points in their time evolution functions. In particular we discussed the models of chaotic dynamical systems in population biology, reactions in chemistry and Edward Lorenz weather system. It is apparent from the paper that dynamical systems can be seen almost everywhere in nature and society. Chaotic dynamical system behavior can be observed in many natural and artificial systems and shall always be a mystery, a paradox, a puzzle, an enigma and a riddle in nature.

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