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***Journal of Scientific and Engineering Research*, 2018, 5(12):317-321**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Monic Gegenbauer Approximations for Solving Differential Equations

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Abstract In this work, we formulate a technique for finding a new method to solve ordinary differential equations (DEs) by using Galerkin spectral method. The Galerkin method depends on Monic Gegenbauer polynomials (MGPs). This work is a direct method, and using equidistance points. Numerical examples are solved to show good ability and accuracy of the present method.

Keywords Monic Gegenbauer polynomials, Differential equations, Spectral Methods, Galerkin method

1. Introduction

To solve DEs, spectral methods were of great importance to this. The unknown solution of the DEs is approximated by a global interplant, such as a polynomial or trigonometric polynomial of high degree. This global interplant is then differentiated exactly, and the expansion coefficients are determined by requiring the equations to be satisfied at an appropriate number of collocation points, see for instance [1-3].

The spectral or pseudospectral formulae method appears as a relation between the expansion coefficients of derivatives and those of the function itself. That is clear, formulae of the expansion coefficients of a general order derivative of finite differentiable orthogonal polynomials in terms of those polynomials themselves are available for Chebyshev [4], Legendre [5], Jacobi polynomials[6] and Gegenbauer polynomials [7-8]. The series of Gegenbauer polynomials are taken our attention. The aim of this work is to present Monic Gegenbauer polynomials and some properties and to obtain MG Galerkin approximation for a given function at the set of equally spaced points.

In [9-11] the authors stated the formula of monic Chebyshev polynomials and used it to solve optimal control problems integral and integro-differential equations. Also, numerical examples are solved to show good ability and accuracy of the monic Chebyshev polynomials.

The organization of this work is as follows: In section 2, we introduce Gegenbauer polynomials and some of its properties. In section 3, we state Monic Gegenbauer polynomials and present the higher derivatives. In section 4, spectral approximation for a given function based on MGPs is presented. Section 5 contained results and discussion. Section 6 is a conclusion.

2. Gegenbauer polynomials and some properties

The Gegenbauer or Ultraspherical polynomials with the real parameter $\left(\lambda > \frac{1}{2},\ \lambda \neq 0\right)$, are a sequence of polynomials $C_j^{\lambda}(x),\quad j = 0,1,2,\cdots$.. For polynomial interpolation to be a well-conditioned process, one must dispense with equally spaced points. As is well known in approximation theory, the right approach is to use point sets that are clustered at the endpoints of the interval with an asymptotic density proportional to $(1 - x2\lambda - 12$ as $n \to \infty$. We use the idea of the approximated function at zeros of the first term of the residual This idea is introduced in the books [12]. In the finite domain $x \in [-1,1]$, each degree j satisfies the orthogonal relation respectively as follows:

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}}\ C_j^{(\lambda)}(x)C_k^{(\lambda)}(x)dx = \begin{cases} 0, & j \neq k \\ \Psi_j^{\lambda}, & j = k \end{cases}, \tag{1}$$

where the normalization constant Ψ_j^{λ} is defined as[13]:

$$\Psi_j^{\lambda} = 2^{1-2\lambda} \frac{\pi\Gamma(j+2\lambda)}{j!(j+\lambda)(\Gamma(\lambda))^2}, \qquad \lambda \neq 0. \tag{2}$$

The polynomials may be generated by following [6]:

$$C_j^{(\lambda)}(x) = \sum_{r=0}^{[j/2]} (-1)^r \frac{\Gamma(j-r+\lambda)}{\Gamma(\lambda)\,(r!)(j-2r)!}(2x)^{j-2r}, \tag{3}$$

where $[j/2]$refers to the integer part of the fraction.

3. Monic Gegenbauer polynomials and some properties

noindent Let $\mathbb{C}_j^{(\lambda)}(x)$ be Monic Gegenbauer polynomials in the finite domain $x \in [-1,1]$, the polynomials may be generated by:

$$\mathbb{C}_j^{(\lambda)}(x) = \mathcal{H}_j^{\lambda} \sum_{r=0}^{\lfloor j/2 \rfloor} (-1)^r \frac{\Gamma(j-r+\lambda)}{\Gamma(\lambda)\Gamma(r+1)\Gamma(j-2r+1)}\ (2x)^{j-2r}, \tag{4}$$

where $\mathcal{H}_j^{\lambda} = 2^{-j} \frac{\Gamma(\lambda)\Gamma(j+1)}{\Gamma(j+\lambda)}$

Hence, the orthogonal relation of MUPs as:

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}}\ \mathbb{C}_j^{(\lambda)}(x)\mathbb{C}_k^{(\lambda)}(x)dx = \begin{cases} 0, & j \neq k \\ \Phi_j^{\lambda}, & j = k \end{cases}, \tag{5}$$

$$\Phi_j^{\lambda} = 2^{1-2\lambda-j} \frac{\pi\Gamma(j+2\lambda)}{\Gamma(j+\lambda+1)\Gamma(\lambda)}, \qquad \lambda \neq 0.$$

Theorem The derivative of order m sense for the MGPs is given by

$$\Phi_j^{\lambda} = 2^{1-2\lambda-j} \frac{\pi\Gamma(j+2\lambda)}{\Gamma(j+\lambda+1)\Gamma(\lambda)}, \qquad \lambda \neq 0.$$

Theorem The derivative of order m sense for the MGPs is given by

$$D^{?}\mathbb{C}_j^{(\lambda)}(x) = \mathcal{H}_j^{\lambda} \sum_{r=0}^{\lfloor (j-m)/2 \rfloor} \Upsilon_{r,j}^{(m)} Q_r^j\ (2x)^{j-2r-m}, \tag{6}$$

where

$$\Upsilon_{r,j}^{(m)} = \prod_{i=0}^{m-1} (j-2r-i), \text{ and } Q_r^j = (-1)^r \frac{\Gamma(j-r+\lambda)}{\Gamma(\lambda)\Gamma(r+1)\Gamma(j-2r+1)}.$$

Proof. We use the mathematical induction to prove the theorem. For $m = 1$, we have

$$\frac{d}{dx}\left[\mathbb{C}_j^{(\lambda)}(x)\right] = \frac{d}{dx}\left[\sum_{r=0}^{\lfloor j/2 \rfloor} \mathcal{H}_j^{\lambda} Q_r^j x^{j-r}\right] = \sum_{r=0}^{\lfloor (j-1)/2 \rfloor} (j-2r)\mathcal{H}_j^{\lambda} Q_r^j x^{j-r}.$$

So the theorem is true for $m = 1$.

Now, assume that the theorem is true for $m = n$.

For $m = n + 1$, we have

$$\frac{d^{n+1}}{dx^{n+1}}\left[\mathbb{C}_j^{(\lambda)}(x)\right] = \frac{d}{dx}\left[\frac{d^n}{dx^n}\left[D^{?}\mathbb{C}_j^{(\lambda)}(x)\right]\right] = \frac{d}{dx}\left(\sum_{r=0}^{\lfloor (j-n)/2 \rfloor} \mathcal{H}_j^{\lambda} \Upsilon_{r,j}^{(m)} Q_r^j x^{j-r-n}\right)$$

$$= \sum_{r=0}^{\lfloor (j-(n+1))/2 \rfloor} (j-r-n)\mathcal{H}_j^{\lambda} \Upsilon_{r,j}^{(n)} Q_r^j x^{j-r-n-1},$$

$$= \sum_{r=0}^{\lfloor (j-(n+1))/2 \rfloor} \mathcal{H}_j^{\lambda} \Upsilon_{r,j}^{(n+1)} Q_r^j\ (2x)^{j-2r-n-1}$$

Hence the theorem is true for every positive integer m.

$$D^m \mathbb{C}_j^{(\lambda)}(x) = \sum_{r=0}^{\lfloor (j-m)/2 \rfloor} \mathcal{H}_j^{\lambda} \Upsilon_{r,j}^{(m)} Q_r^j\ (2x)^{j-2r-m}.$$

4. Monic Gegenbauer Galerkin Spectral Method (MGGM)

This part contains information about the method to solve some types of linear or nonlinear ODEs.

Consider the function $F(x)$ is an integrable square function on $[-1,1]$. So, it can be as a form of series in terms of MGPs:

$$F(x) = \sum_{j=0}^{\infty} a_j \mathbb{C}_j^{(\lambda)}(x), \tag{7}$$

where a_j are arbitrary constant.

Now, assume that the linear DEs as:

$$D^m F(x) + \sum_{i=0}^{r-1} \rho_i(x) D^{s_i} F(x) + g(x)F(x) = G(x). \tag{8}$$

Such as the boundary conditions

$$F^{(k)}(1) = d_k, \quad F^{(l)}(-1) = e_l, \quad 0 \le i, k \le m-1. \tag{9}$$

Let m, r, s_i, d_k, e_l, k, and i are integer number, $s_i < m$, $and\ i$, r, $s_i \ge 0$, and $m \ge 1$. Also, $\rho_i(x)$, $g(x)$, and $G(x)$ are function of x.

Hence, we define

$$\begin{aligned} &x = (x_0,\ x_1,\ \ldots, x_N)^T, \\ &S_N(I) = span\left\{C_0^{(\alpha)}(x), C_1^{(\alpha)}(x),\ \cdots, C_N^{(\alpha)}(x)\right\}, \\ &\rho_i(x) = \left(\rho_i(x_0),\ \rho_i(x_1),\ \ldots,\ \rho_i(x_N)\right)^T, \\ &g(x) = \left(g(x_0),\ g(x_1),\ \ldots,\ g(x_N)\right)^T, \end{aligned} \tag{10}$$

Dente that a_j in equation 7 can be $\boldsymbol{a} = (a_0,\ a_1,\ \ldots, a_N)^T$.

Now, by using equations 6 and 7 to get:

$$D^m F(x) = F^{(m)}(x) = \frac{d^q}{dx^q} \sum_{j=0}^{N} a_j \mathbb{C}_j^{(\lambda)}(x), \tag{11}$$

At $m = 0$ equation 11 will be same equation 7.

By the same substitution at the boundary condition:

$$\sum_{j=0}^{N} a_j \left(\mathbb{C}_j^{(\lambda)}(1)\right)^k = d_k, \quad \sum_{j=0}^{N} a_j \left(\mathbb{C}_j^{(\lambda)}(-1)\right)^l = e_l, \quad 0 \le i, k \le m - 1.4$$

By apply the relations 6 and 7 in 8, to get the following result:

$$D^m \sum_{j=0}^{N} a_j \mathbb{C}_j^{(\lambda)}(x) + \sum_{i=0}^{r-1} \rho_i(x) D^{s_i} \sum_{j=0}^{N} a_j \mathbb{C}_j^{(\lambda)}(x) + g(x) \sum_{j=0}^{N} a_j \mathbb{C}_j^{(\lambda)}(x) = G(x). \tag{12}$$

Equation 12 will be written as:

$$\sum_{j=0}^{N} a_j \left(\mathbb{C}_j^{(\lambda)}(x)\right)^m + \sum_{i=0}^{r-1} \rho_i(x) \sum_{j=0}^{N} a_j \left(\mathbb{C}_j^{(\lambda)}(x)\right)^{s_i} + g(x) \sum_{j=0}^{N} a_j \mathbb{C}_j^{(\lambda)}(x) = G(x). \tag{13}$$

We can take a_j a common factor from equation 13

$$a_j \left[\sum_{j=0}^{N} \left(\mathbb{C}_j^{(\lambda)}(x)\right)^m + \sum_{i=0}^{r-1} \rho_i(x) \sum_{j=0}^{N} \left(\mathbb{C}_j^{(\lambda)}(x)\right)^{s_i} + g(x) \sum_{j=0}^{N} \mathbb{C}_j^{(\lambda)}(x)\right] = G(x). \tag{14}$$

From equation 10 where $x = x_n$, $n = 0,1,\cdots,N$ represent by span of N equation 14 will be:

$$a_j \left[\sum_{j=0}^{N} \left(\mathbb{C}_j^{(\lambda)}(x_n)\right)^m + \sum_{i=0}^{r-1} \rho_i(x_n) \sum_{j=0}^{N} \left(\mathbb{C}_j^{(\lambda)}(x_n)\right)^{s_i} + g(x_n) \sum_{j=0}^{N} \mathbb{C}_j^{(\lambda)}(x_n)\right] = G(x_n). \tag{15}$$

Now, equations (15) represent a system of equation in a_j unknowns. These equations are being linear or nonlinear equations depend on type of equation (8). Where equations (15) are followed equation (8) in being linear or nonlinear equations. By convert equations (15) to matrix $(N+1) \times (N+1)$ can be solved too.

5. Result of Examples

This part consists of analyzed examples. The given results are to be compared by other methods'.

Example 1 Consider the following linear third-order boundary value problem

$$8y^{(3)}(x) - 0.5(x+1)y(x) = (0.125(x+1)^3 - 0.5(x+1)x^2 - 2.5(x+1) - 3e0.5(x+1), \quad -1 \le x \le 1 \tag{16}$$

subject to the boundary conditions

$$y(-1) = 0, \quad y(1) = 0, \quad y'(-1) = 0.5, \tag{17}$$

with the analytic solution $y(x) = 0.25(1 - x^2)e^{(0.5(x+1))}$.

Table (1) describe a maximum absolute error for Example 1. This table contain approximate solution by using different values of N and λ. Those results are compared by another way. Also, in [15] the best MAE is 1.13 e−10, Plus this method needs many steps compared to our method.

Table 1: The MAEs by using MGGM for Example (1)

N	λ =0.49	λ =-0.49	λ =0.5	λ =1	[14]	[11]
4	2.22130e-16	0	1.29586e-16	1.5803e-16	1.00234e−6	-
6	6.4840e-16	2.0452e-17	1.13130e-16	8.3127e-16	8.44313e−10	2.563e-04
8	4.3215e-16	2.3620e-16	8.2684e-16	1.1193e-16	5.05134e−13	-
10	6.1786e-12	7.0882e-15	2.0977e-14	2.01587e-14	4.92184e−16	9.966e-10

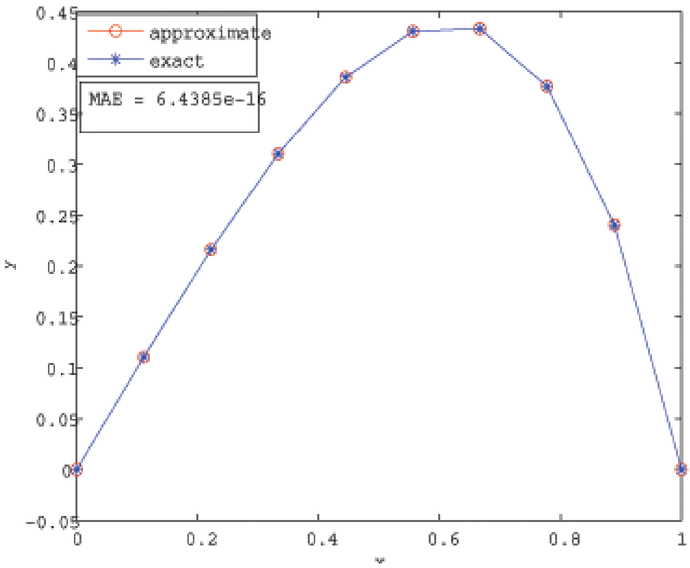

Figure 1: The exact and approximate solution at N=9 and λ =0.49 for Example 1

6. Conclusion

This work proved high accuracy and efficiency. Plus it uses a direct method. Monic Gegenbauer polynomials is considered an important polynomials where is form a large family of orthogonal polynomials. Our method can be applied at different types of ordinary differential equations.

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