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Research Article

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A Numerical Solution of the 2D Laplace's Equation for the Estimation of Electric Potential Distribution

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Abstract Many applications in Science and Engineering have found Laplace's equation very useful. A numerical solution of the equation can be useful in finding the distribution of temperature in a solid body, the potential distribution in a region of interest and so on. In this work, Finite Difference Method (FDM) was used to discretize Laplace's equation and then the equation was solved numerically using three different iterative methods with the application of different Dirichlet boundary conditions. The iterative methods used include the Jacobi, the Gauss Seidel and the Successive Over Relaxation (SOR) methods. The results obtained indicates that no local minima or maxima were observed in the distribution of electric potential in the square grid region. The results were compared based on the nature of Dirichlet boundary conditions and it was observed that, the pattern of the potential distribution depended greatly on the nature of the boundary conditions. It was observed finally by the comparison of the three iterative methods that SOR method is the most effective in terms of accuracy and speed of convergence.

Keywords Laplace's equation, Dirichlet boundary condition, Iterative Methods, Minima, Maxima

1. Introduction

Laplace's equation is an important equation that has been associated with so many applications in the field of Science and Engineering. These applications are not only limited to electrostatics, but fluid dynamics as well as steady state heat conduction too. It is a partial differential equation (PDE) of second order which is elliptic in nature [1]. A general theory of the solution of Laplace's equation according to [2] is called potential theory. The solutions of Laplace's equation are usually known as harmonic functions as can be seen in [3-5] and are very useful in science and engineering as earlier mentioned. Laplace's equation has both analytic and numerical solutions. Numerical solution of Laplace's equation is obtained by different methods as applied to many linear PDEs. These include finite difference method (FDM), finite element method (FEM) as well as the Method of Moments (MoM) as found in [6] and also, Markov Chain method (MCM) covered in [7]. Our approach to solve Laplace's equation numerically focuses on FDM.

Laplace's equation itself does not determine the potential of a system, thus, suitable boundary conditions must be applied to obtain the potential of any system of interest [5]. A boundary condition can be Dirichlet if it is specified at the surface of the boundary of a system, it can be Neumann if the normal derivative of the function is specified at the boundary or it can be mixed if part of the boundary is Dirichlet and the rest Neumann [8].

Laplace's equation has received attention from many Researchers for different physical applications. A numerical solution of Laplace's equation was obtained in [9] where he compared the result with analytic solution and found that, the results were similar. Patil and Prasad solved Laplace's equation numerically using FDM, FEM and MCM [7]. They compared the results with analytic solution and observed that the solution of each method used was in close agreement with the exact solution. These two works indicate that, numerical solution of Laplace's equation is in good agreement with its analytic solution. The effectiveness of Successive

over relaxation in numerical solution has been emphasized by many researchers as can be seen in [10], [11], [12] & [13]. However, the reviewed literatures so far were not able to bring out a clear comparison between these iterative methods. No knowledge of how the potential distribution depends clearly on the nature of Dirichlet boundary conditions has been unveiled. All the mentioned gaps motivated this research work.

Thiswork focuses on the estimation of the distribution of electric potential through numerical solution of 2-Dimensional Laplace's equation in cartesian coordinate. Finite difference method (FDM) was explored to achieve this purpose by utilizing its different iterative schemes. These schemes include Jacobi iteration, Gauss – Seidel iteration and Succesive Over Relaxation (SOR) iteration schemes. This was done to check the effectiveness of each of the iteration scheme. Also, the dependence of the electric potential distribution in the square grid region on the nature of the Dirichlet boundary conditions applied was investigated.

The remaining part of the paper is organized as follows; section 2 presents materials and methods used in this research. Details of Laplace's equation and solution are presented here. Results are presented and discussed in section 3 and section 4 concludes the research.

2. Materials and Methods

According to [5], Gauss' law may be written as:

$$\nabla . \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad (1)$$

The electric field intensity can be expressed in terms of scalar potential as:

$$\mathbf{E} = -\nabla V \qquad (2)$$

Equation (2) substituted into equation (1) results to:

$$\nabla^2 \mathrm{V} = -\frac{\rho}{\varepsilon_0} \qquad (3)$$

Where V is electric potential, ρ is charge density and ε_0 is the permittivity of free space. Equation (3) above is called Poisson's equation.

The interest of this work is to find the potential in a charge free region (i.e.$\rho = 0$). Therefore, poison equation reduces to Laplace's equation written as:

$$\nabla^2 \mathrm{V} = 0 \qquad (4)$$

Laplace's equation in 3-D Cartesian coordinates is expressed as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (5)$$

However for 2-D, Laplace's equation is given as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (6)$$

Finite difference Method (FDM) and Discretization of Laplace's Equation

To compute the potential from Laplace's equation, the equation is discretized with constant grid spacing in the x and y directions respectively. A rectangular body whose boundaries are conducting surfaces is divided into grid points of $m \times n$constant spacing and Dirichlet boundary conditions are applied. The grid spacing in the x direction is denoted by h_x while that in the y direction is denoted by h_y. Since the grid point spacing is constant, $h_x = h_y = h$.

The grid points are indexed by (k, l), where $k, l = 1, 2, 3, \ldots, m, n$. Thus, the finite difference formula is given by [8]:

$$\frac{\partial^2 V}{\partial x^2} \approx \frac{V_{k+1,l} - 2V_{k,l} + V_{k-1,l}}{h^2} \qquad (7)$$

and

$$\frac{\partial^2 V}{\partial y^2} \approx \frac{V_{k,l+1} - 2V_{k,l} + V_{k,l-1}}{h^2} \qquad (8)$$

where h is the constant spacing on the grid points.

Equations (7) and (8) are substituted in (6) which produce:

$$V_{k+1,l} + V_{k-1,l} + V_{k,l+1} + V_{k,l-1} - 4V_{k,l} = 0 \qquad (9)$$

Equation (9) is the discretized Laplace's equation. Rearranging it leads to:

$$V_{k,l} = \frac{1}{4}\left[V_{k+1,l} + V_{k-1,l} + V_{k,l+1} + V_{k,l-1}\right] \qquad (10)$$

Equation (10) implies that, every potential grid point depends on the values of the four nearest neighboring potential grid points as can be illustrated in a five point stencil below:

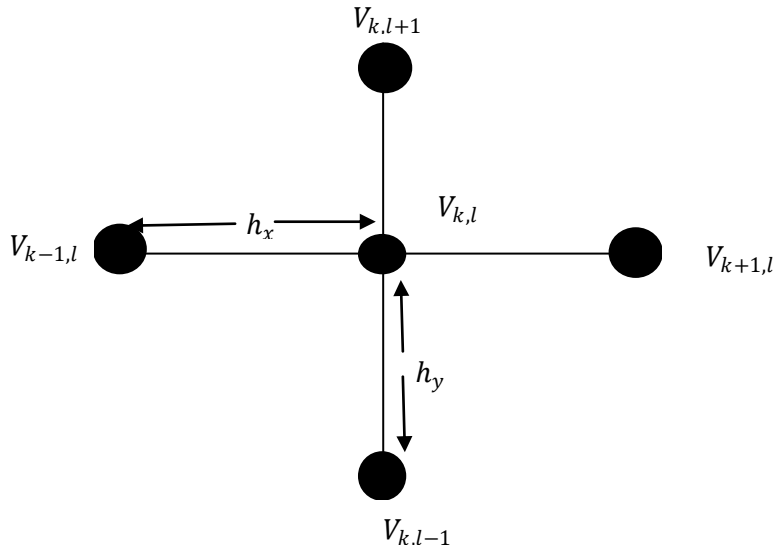


Figure 1: Five Point Stencil for Potential Computation

This can be presented in a generalization of grid point for computation of electric potential as shown below:

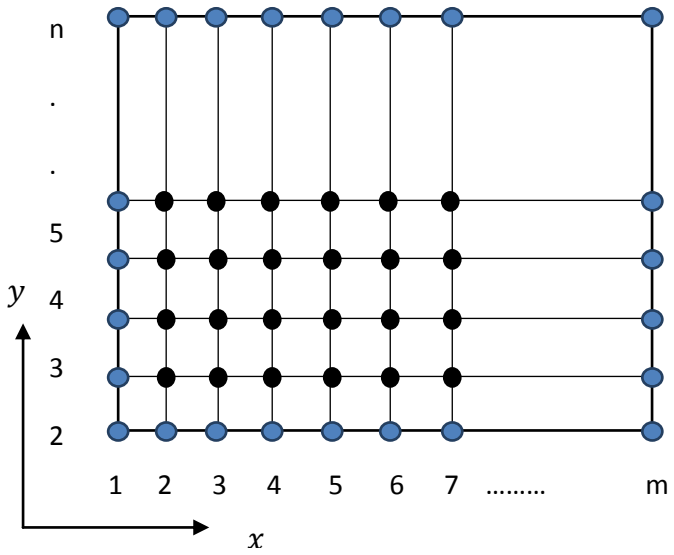


Figure 2: Generalize Grid Point for Potential Computation

As can be seen from the diagram above, the blue grid points at the boundaries correspond to the boundary values which are represented according to equation (10) by the potential values $V_{1,1}, V_{2,1}, \ldots, V_{m,1}$ from the bottom across the grid points, $V_{1,n} V_{2,n}, \ldots, V_{mn}$ from the top across the grid points, $V_{1,1}, V_{1,2} \ldots V_{1,n}$ from the bottom left to the top of the grid points and $V_{m,1}, V_{m,2}, \ldots, V_{mn}$ from the bottom right to the grid of the points. The potential values to be evaluated are those in the interior grid points which sweep across the grid points from $V_{2,2}, V_{3,2}, V_{4,2} \ldots V_{m-1,2}$ up to $V_{2,n-1}, V_{3,n-1}, \ldots, V_{m-1,n-1}$

A system of equation, one for each potential node as generated from equation (10) is shown below

$$\left.\begin{aligned}
V_{2,2} &= \frac{1}{4}\left[V_{1,2} + V_{3,2} + V_{2,3} + V_{2.1}\right] \\
V_{3,2} &= \frac{1}{4}\left[V_{2,2} + V_{4,2} + V_{3,3} + V_{3,1}\right] \\
&\vdots \\
V_{m-1,2} &= \frac{1}{4}\left[V_{m-2,2} + V_{m,2} + V_{m-1,3} + V_{m-1,1}\right] \\
V_{2,3} &= \frac{1}{4}\left[V_{1,3} + V_{3,3} + V_{2,4} + V_{2,2}\right] \\
V_{3,3} &= \frac{1}{4}\left[V_{2,3} + V_{4,3} + V_{3,4} + V_{3,2}\right] \\
&\vdots \\
V_{m-1,3} &= \frac{1}{4}\left[V_{m-2,3} + V_{m,3} + V_{m-1,2} + V_{m-1,4}\right] \\
V_{2,n-1} &= \frac{1}{4}\left[V_{1,n-1} + V_{3,n-1} + V_{2,n-2} + V_{2,n}\right] \\
V_{3,n-1} &= \frac{1}{4}\left[V_{2,n-1} + V_{4,n-1} + V_{3,n-2} + V_{3,n}\right] \\
&\vdots \\
V_{m-1,n-1} &= \frac{1}{4}\left[V_{m-2,n-1} + V_{m,n-1} + V_{m-1,n} + V_{m-1,n-1}\right]
\end{aligned}\right\} \qquad (11)$$

A solution of the above system of equation may be obtained from direct Gaussian elimination method for a system of small $(m \times n)$ unknown. For a large system of unknown, iterative methods achieve a better result.

For all grid based numerical schemes according to [14], the accuracy of the numerical results depends greatly on the computational grid. Thus, a grid converged solution would be preferred for accuracy (i.e. when more grid point are used, solution does not change significantly as one approaches a tolerance point). Three iterative methods are proposed to use for this work which include; the Jacobi method, the Gauss – Seidel method and the Successive over Relaxation method (SOR). Details of each iterative method is provided below

The Jacobi Iteration Method

The Jacobi iterative formula is given by [15]:

$$\mathrm{V}_{k,l}^{q+1} = \frac{1}{4}\left[V_{k-1,l}^{q} + V_{k+1,l}^{q} + V_{(k,l-1)}^{q} + V_{k,l+1}^{q}\right] \qquad (12)$$

The superscript q is an iterative index. The initial iterative guess can be set at $q = 0$, to produce $V_{k,l}^{0}$ and then successively improve it according to the iteration. From equation (12) above, the next iteration (q+1) can be found for each grid point (k, l) across all the grid points in the horizontal rows. On completion of the iteration for all interior grid points, the difference between the vectors of the next iteration V^{q+1} and the previous iteration V^{q} is computed. The iteration terminates once the predefined condition (tolerance) set for the iteration to converge is met and the solution to (12) is V^{q+1}, otherwise the iterations continue. i.e.

$$|V^{q+1} - V^{q}| < tolerance$$

The Gauss–Seidel Iteration Method

The Gauss-Seidel iteration formula is given in [16] by;

$$\mathrm{V}_{k,l}^{q+1} = \frac{1}{4}\left[V_{k-1,l}^{q+1} + V_{k+1,l}^{q} + V_{k,l-1}^{q+1} + V_{k,l+1}^{q}\right] \qquad (13)$$

As can be seen from equation (13), the values of $V_{k-1,l}$and $V_{k,l-1}$updated already as one moves through the grids to reach the grid point (k, l).The implementation of this iteration scheme is the same as in Jacobi scheme.

The Successive Over-Relaxation (SOR) Method

This is the most used iteration method and is embedded in the Gauss-Seidel Method. A relaxation parameterλ is included in the Gauss-Seidel iteration formula with the aim of quickening the convergence. This is given according to [17] by;

$$V_{k,l}^{q+1} = (1-\lambda)V_{k,l}^{q} + \frac{\lambda}{4}\left[V_{k-1,l}^{q+1} + V_{k+1,l}^{q} + V_{k,l-1}^{q+1} + V_{k,l+1}^{q}\right] \qquad (14)$$

The relaxation parameter is of the range $1 < \lambda < 2$ having a optimal value given by $\lambda = 2 - \pi h$ [16].Implementation of SOR proceeds the same way as the first two methods.

MATLAB programs are developed for the three iteration methods and Dirichlet boundary conditions are applied for a 100×100 grid. The first boundary condition is applied in such a way that, the three boundaries are grounded, which is the left, right and the bottom boundaries while the top boundary is set at a potential of 1.0 $volt$. In the second boundary condition, the left and right are grounded while the bottom and top boundaries are maintained at a potential of 1.0 $volt$. This can be written mathematically as:

$$V(0,y) = 0, \quad V(a,y) = 0, \qquad V(x,0) = 0 \; and \; V(x,b) = 1.0$$

And

$$V(0,y) = 0, \quad V(a,y) = 0, \qquad V(x,0) = 1.0 \; and \; V(x,b) = 1.0$$

The tolerance is set at

$$tolerance = 10^{-12}$$

3. Results and Discussions

This section presents the results of the computed electric potential based on the iteration schemes and the boundary conditions discussed above.

Fig. 3 presents the results of electric potential distribution as a numerical solution of Laplace's equation with Dilichlet boundary condition applied such that the left, right and bottom boundaries are grounded while the top boundary is maintained at a potential of 1.0 $Volt$. The results obtained through the different iteration schemes exhibit similar pattern in their surface plots of Fig. 3 (a), (b) and (c) respectively. As it is the case of the solution of Laplace's equation, the electric potential increases from the left to the middle and decreases to the right. It also increases steadily from the bottom to the top. This clearly indicates that there are no local maxima or minima found in the solution. According to [5], it is a requirement of Laplace's equation that, as the charge density vanishes at the second derivative, it is reasonable that the solutions show no maxima or minima.

Despite the fact that the plots of the solution exhibit similarities with different iteration schemes, there exist differences in terms of accuracy and convergence between different iteration schemes. The Jacobi iteration method based on the tolerance set converged very slowly with an error limit of $9.9948 \times 10^{-13}$and 24293 number of iterations. The Gauss Seidel iteration method converged faster than the Jacobi iteration method with 12811 number of iterations but an error limit of 9.9953×10^{-13}. This clearly indicates that, the improvement of Gauss Seidel method over Jacobi method is only in the reduction of the number of iterations as you do not need many iterations to arrive the result. There is no significant difference in their accuracy. Successive Over Relaxation method proved to be the most powerful and efficient iteration scheme than the previous iteration methods. SOR method converged at an error limit of 2.9060×10^{-13} and with 572 number of iterations. This is far better than the previous methods. The relaxation parameter indeed played an important role in speeding up the iterations.

The plot of error against the number of iterations for the three iteration methods are shown in Fig. 4. The plots show how the various iteration methods converged as the number of iterations increased. It is obvious from the graphs that, the Jacobi iteration method converged slower than the other methods while SOR converged faster than all of them and is also more accurate.

It is also important to know that, the number of iterations is proportional to number of grid points considered in the region irrespective of the iteration method used.

In figure 5, the result of electric potential distribution in a square grid system is presented. Dirichlet boundary condition applied here is such that, the left and the right boundaries of the grid are grounded while the bottom and the top boundaries are maintained at 1.0 $Volt$. Having established the efficiency of the SOR from the above comparison, the computation of the electric potential was done using SOR method. The result show increase in electric potential from the left to the middle and then decreases to the right, decreases from the bottom of the grid to the middle and again increases to the top.

The electric potential in the middle of the grid appears to be the maximum potential for electric potential distributed from the left to the right and appears to be the minimum potential for that distributed from the bottom to the top. However, this point is neither a local minimum nor a local maximum. This again agrees with Griffiths' assertion that, local minima or maxima does not exist in the solution of Laplace's equation [5]. It is clear from this result that, the distribution of electric potential in the region of a square grid system depends on the nature of the Dirichlet boundary conditions applied.

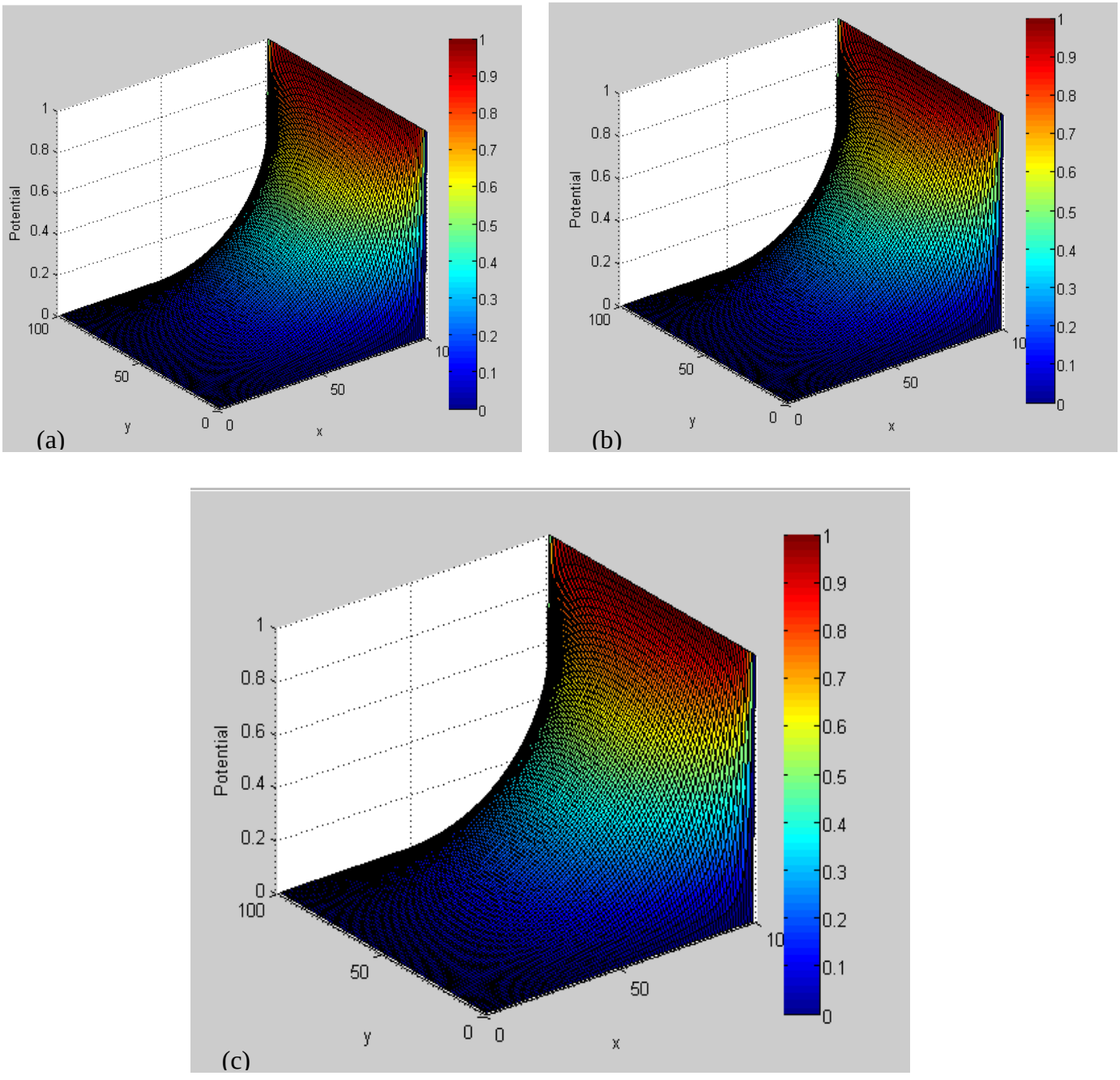


Figure 3: Surface Plot of Potential Distribution through (a) Jacobi Iteration Method, (b) Gauss Seidel Iteration Method and (c) Over Relaxation Iteration Method.

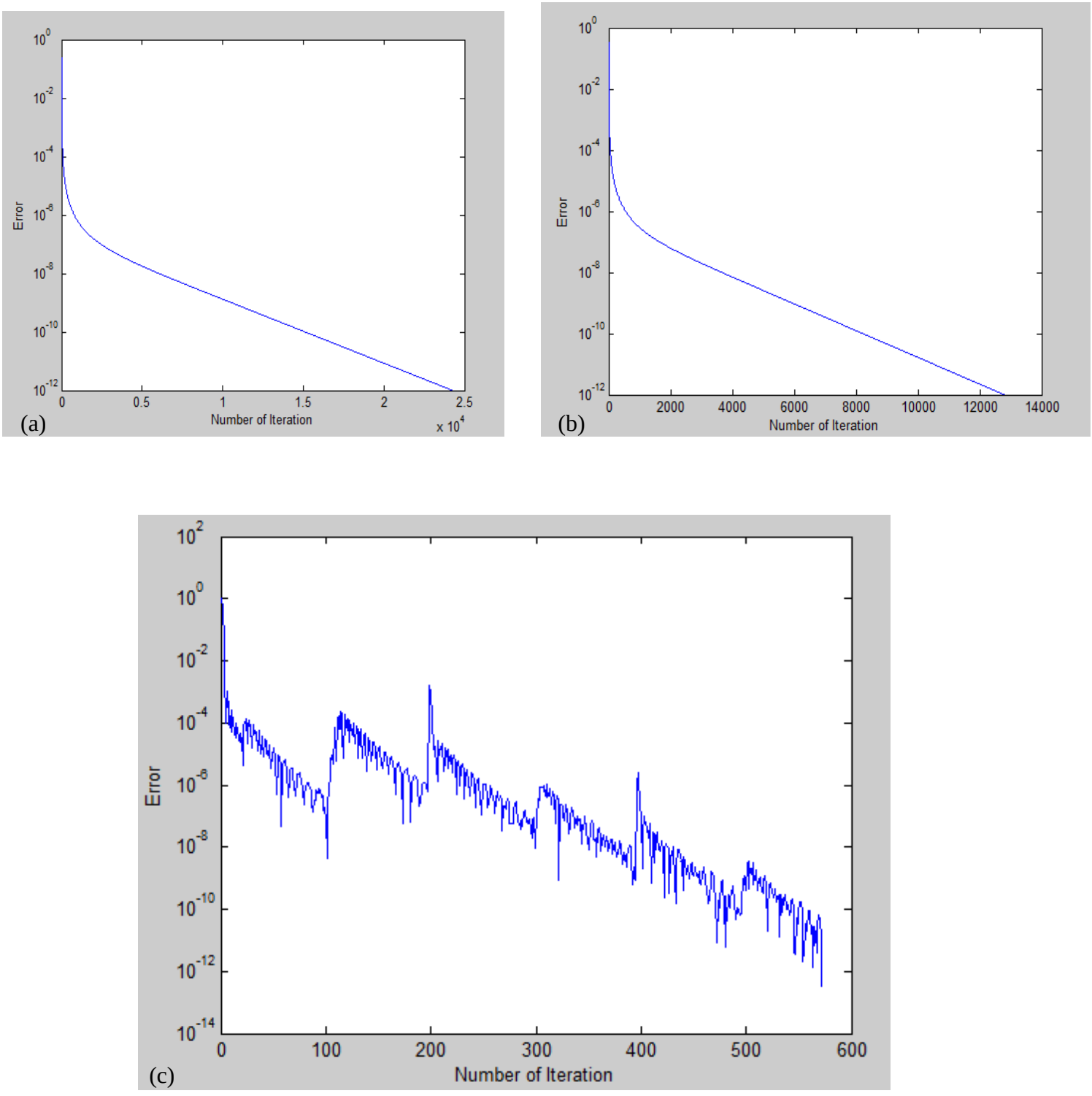
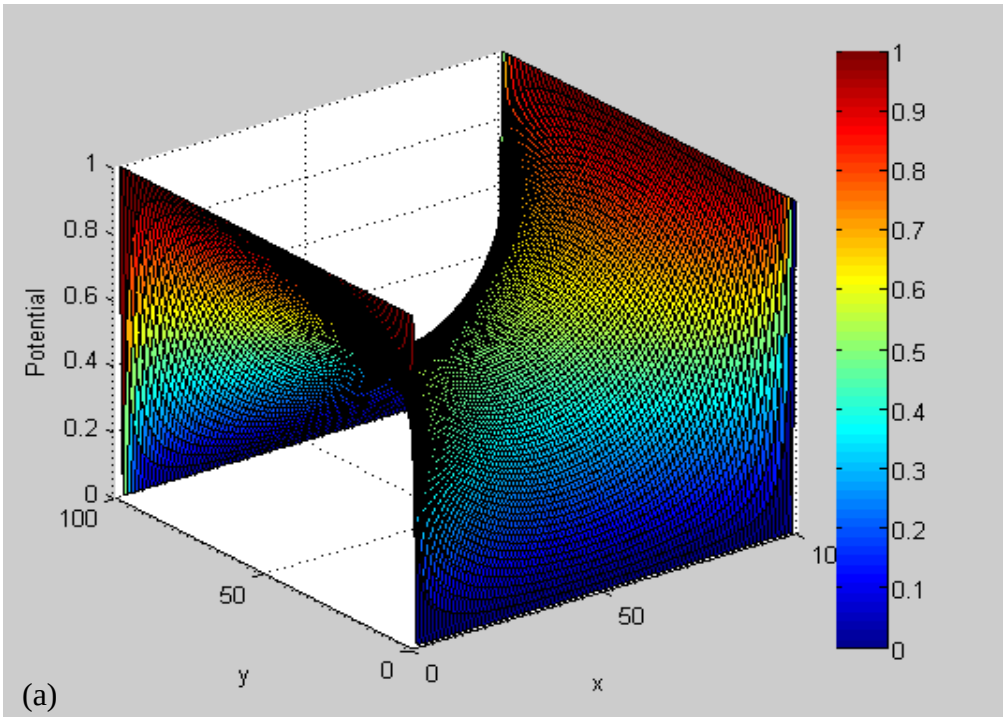


Figure 4: Error Plot for (a) Jacobi Iteration Method (b) Seidel Iteration Method and (c) Successive Over Relaxation Iteration Method



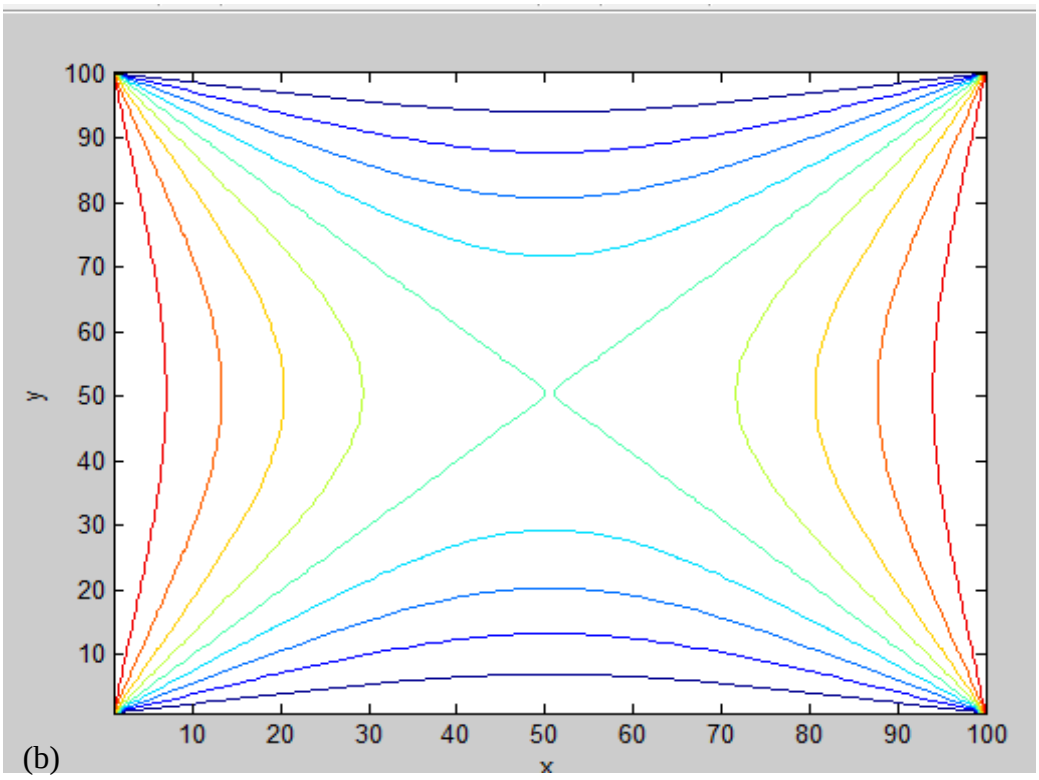


Figure 5: (a) Surface Plot of Electric Potential Distribution (b) Contour Plot of Electric Potential Distribution

4. Conclusion

This research considered the distribution of electric potential in a square grid region through the numerical solution of Laplace's equation. Three different iteration schemes under Finite Difference Method (FDM)were compared which are the Jacobi, the Gauss Seidel and the SOR iteration methods. The results obtained showed clearly that, the SOR iteration method is more efficient in terms of accuracy and speedy convergence. Two different Dirichlet boundary conditions were also considered which showed different patterns in the distribution of the electric potential in a square grid region. This indicated that the distribution of electric potential in a square grid region depend on the nature of the Dirichlet boundary conditions applied. In each case, it was observed that the potential distribution exhibited no local minima or maxima.

References

[1]. Urroz G. E. Introduction to Partial Differential Equations. 2004

[2]. Moradi G. and Marvasti M. Experimental Solution to the Laplace Equation, a Tutorial Approach. International Journal of Advanced Research in Computer and Communication Engineering. 2016; 5(9):278-284.

[3]. Morales M., Diaz. R. A., Herrera W. J. Solutions of Laplace's equation with simple boundary conditions, and their applications for capacitors with multiple symmetries. arXiv:1503.09183v1 [physics.class-ph]. 2015.

[4]. Prilenko A. I., and Solomentsev E. D. "Potential theory", Encyclopedia of Mathematics. 2001.

[5]. Griffiths D. J. Introduction to Electrodynamics. (3rd Ed). Prentice – Hall Inc., Upper Saddle River, NewJersey. 1999; 110 – 159.

[6]. Sadiku M. N. O. "Numerical Technics in Electromagnetics", CRC Press. 2000.

[7]. Patil P. V., and Prasad J.S.V.R. K. Numerical Solution for Two Dimensional Laplace Equation with Dirichlet Boundary Conditions. IOSR Journal of Mathematics (IOSR-JM).2013; 6(4):66-75

[8]. Dhumal, M. L. and Kiwne S. B. Finite Difference Method for Laplace Equation. International Journal of Statistika and Mathematika. 2014; 9(1):11-13

[9]. Guthrie M. Solving Laplace's Equation with MATLAB Using the Method of Relaxation. 2010.

[10]. Mai, T. Z., and Wu, L. The Successive over Relaxation Method in Multi-layer Refinement Scheme. Advances in Applied Science Research. Pelagia Research Library, CODEN (USA). 2013; 4(2):163-168.

[11]. Justine, H. and Sulaiman J. Solution of Fourth – Order Two – Point BVPs with Cubic Non – Polynomial Spline and SOR iterative Method. J. Fundam. Appl. Sci. 2017; 9(5):579 – 593.

[12]. Mayooran, T. and Light, E. Applying the Successive Over relaxation Method to a Real World Problems. American journal of Applied Mathematics and Statistics. 2016; 4(4):113 – 117

[13]. Lee, P. Kim, T. W. and Kim, S. Accurate and Efficient Numerical Solutions for Elliptic Obstacle Problems. Journal of Inequalities and Applications. DOI.10.1186/s13660 – 017 – 1309 – 2. 2017.

[14]. Mitra, A. K. Finite Difference Method for the solution of laplace equation. American Journal of Physics. 2011; 1-13.

[15]. Causon, M and Mingham, G. Introductory Finite Difference Methods for PDES. Delhi: Ventus Publishing APS. 2014.

[16]. Vendromin, C. Numerical Solution of laplace Equation for Various Physical Situations. Brock University, St. Catharines canada. 2017; 77-89.

[17]. Yang, X. and Mittal, R. Acceleration of the Jacobi Iterative Method by factors exceeding 100 using Scheduled Relaxation. Journal of Computational Physics. 2014; 274:695 – 708.