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Research Article

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Solutions of non Linear Fractional Partial Differential Equations by Using Sub-Equation Method

M. K. Ammar, Amany Saad, Nahla Bahaa

Department of Mathematics, Helwan University, Cairo, Egypt

Abstract In this paper we introduce the sub-equation method to solve nonlinear fractional partial differential equations in two applications namely the space time fractional Potential Kadomstev – Petviashvili (PKP) equation and Burgers' equation.

Keywords Sub-equation method, nonlinear fractional partial differential equations,space time fractional Potential Kadomstev – Petviashvili (PKP) equation and Burgers' equation

1. Introduction

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation . Fractional differential equations, therefore find numerous applications in the field of viscoelasticity, feed- back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences. Many strategies for tackling fractional differential equations were proposed, such asAdomian decomposition method [1-2], complex transform method [3-4], exponential function method [5-6], the first integral method [7-8]. Many problems used the sub equation method [9-10], and we solve the nonlinear fractional partial differential equations by using this method. Most recently, according to homogeneous balance principle and Jumaries modified Riemann-Liouville derivative, Zhang and Zhang presented a novel technique, that is fractional sub-equation method. A fractional sub-equation method is proposed to solve fractional differential equations.

2. Modified Riemann–Liouville Derivative

Modified Riemann – Liouville derivative of order β is defined by [11]

$$D_z^{\beta} h(z) = \frac{1}{\Gamma(1-\beta)} \int_0^z (z-\zeta)^{-\beta-1} (h(\zeta)-h(0)) d\zeta, \beta < 0 \tag{1}$$

$$= \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z (z-\zeta)^{-\beta} (h(\zeta)-h(0)) d\zeta, 0 < \beta < 0 \tag{2}$$

$$= [h^{\beta-n}(z)]^n, 0 \le \beta < n+1,\ n \ge 1 \tag{3}$$

The properties of modified Riemann – Liouville derivative which are used in this paper [12-13]

$$D_z^{\beta} z^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)} z^{\gamma-\beta}, \gamma > 0 \tag{4}$$

$$D_z^{\beta} (h(z)\ g(z)) = g(z)\ D_z^{\beta} h(z) + h(z)\ D_z^{\beta} g(z) \tag{5}$$

$$D_z^{\beta} f[g(z)] = f'[g(z)]\ D_z^{\beta} g(z) = D_z^{\beta} f[g(z)] g'(z)^{\beta} \tag{6}$$

3. The sub–Equation method

This method is based on four steps for solving the fractional partial differential equations

Let

$$u(V,\ V_t, V_x, D_t^{\beta}V, D_x^{\alpha}V, D_{tt}^{2\beta}V,\ D_{xx}^{2\alpha}V, \dots) = 0, 0<\beta<1, 0<\alpha<1 \tag{7}$$

be a nonlinear fractional partial differential equations in x and t.

3.1. The first step

Let

$$V(x,t) = V(\zeta), \zeta = \frac{\ell x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\lambda t^{\beta}}{\Gamma(1+\beta)} \tag{8}$$

where ℓ and λ are constants. Then (7) will transform to:

$$u(V,\ V',\ V'', \dots) = 0 \tag{9}$$

where the symbol " $'$ " represents the derivative w. r. to $.\zeta$

3.2. The second step

Suppose that (9) has the solution

$$V(\zeta) = \sum_{m=0}^{n} C_m\ \psi^m \tag{10}$$

Where $C_m\ (m = 0,1,\dots,n)$ are constants, we will evaluate them, n is determined by the balancing in eq. (9) between the highest order derivative and the nonlinear term. Let $\psi = \psi(\zeta)$ be the solution of Riccati equation

$$D'\psi = \sigma + \psi^2 \tag{11}$$

where σ is a constant, so eq. (11) has the solutions 141516

$$\psi(\zeta) = \begin{cases} = -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\zeta), & \sigma<0 \\ = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\zeta), & \sigma<0 \\ = \sqrt{\sigma}\tan(\sqrt{\sigma}\zeta), & \sigma>0 \\ = -\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta), & \sigma>0 \\ = -\frac{1}{\zeta+w},\quad w = const, & \sigma = 0 \end{cases} \tag{12}$$

3.3. The third step

Substitute eq. (10) into eq. (9) and using (11). We obtain a polynomial in $\psi(\zeta)$. Equate the coefficients of the same power of ψ^m, we have nonlinear equations of $C_m\ (m = 0,1,\dots,n)$

3.4. The fourth step

Solve these equations by using Mathematica program, and substitute these solutions in eq. (10)

4. Applications

We evaluate the solutions of some nonlinear fractional partial differential equations by using the sub-equation method.

Example 1 *Burgers' equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. It is named for Johannes Martinus Burgers* $(1895-1981)$.*For a given field* $y(x,t)$ *and diffusion coefficient (or viscosity, as in the original fluid mechanical context), the general form of Burgers' equation (also known as viscous Burgers' equation) in one space dimension is the dissipative system. This equation has the form* 8

$$D_t^{\beta}V + V\ D_x^{\beta}V = \nu\ D_x^{2\beta}V, 0<\beta\le 1,\quad t>0 \tag{13}$$

$$let V(x,t) = V(\zeta), \qquad \zeta = \frac{\ell x^{\beta}}{\Gamma(1+\beta)} + \frac{\lambda t^{\beta}}{\Gamma(1+\beta)} \tag{14}$$

Where $\ell,\ \lambda$ *and* ν *are constants then eq.* (20) *tansforms to*

$$\lambda V' + \ell V\ V' - \nu\ \ell^2 V'' = 0$$

Integrate this equation, we obtain

$$\lambda V + \frac{1}{2}\ell V^2 - \nu\ \ell^2 V^{'} = 0 \tag{15}$$

By balancing the nonlinear term and the highest order derivative, we obtain $n = 1$, so eq. (22)has the solution

$$V(\zeta) = C_0 + C_1\psi \tag{16}$$

And the derivatives of V are

$$V^{'}(\zeta) = C_1\sigma + C_1\psi^2$$

Substitute these equations in eq. (15)

$$\psi^2:\ -\nu C_1\ell^2 + \frac{1}{2}\ell\ C_1^2 = 0$$

$$\psi^1:\ \lambda C_1\ + \ell C_0\ C_1 = 0$$

$$\psi^0:\ \lambda C_0\ + \frac{1}{2}\ell\ C_0^2 - \nu C_1\ell^2\sigma = 0$$

We solve this system by using Mathematica Program, we obtain

$$C_0 = -\frac{\lambda}{\ell}, \qquad C_1 = 2\ \nu\ \ell, \qquad \sigma = -\frac{\lambda^2}{4\nu^2\ \ell^4} \tag{17}$$

So the solutions of eq. (13) by using (12,14,16,17)are:

$$V_1(x,t,y) = -\frac{\lambda}{\ell} + \frac{\lambda}{\ell}\tanh\left(\frac{\lambda}{2\nu\ \ell^2}\left(\frac{\ell x^\beta}{\Gamma(1+\beta)} + \frac{\lambda t^\beta}{\Gamma(1+\beta)}\right)\right) \tag{18}$$

$$V_2(x,t,y) = -\frac{\lambda}{\ell} - \frac{\lambda}{\ell}\coth\left(\frac{\lambda}{2\nu\ \ell^2}\left(\frac{\ell x^\beta}{\Gamma(1+\beta)} + \frac{\lambda t^\beta}{\Gamma(1+\beta)}\right)\right) \tag{19}$$

Example 2 *The space time fractional potential Kadomstev – Petviashvili (PKP) equation* 17

$$\frac{1}{4}D_t^{4\beta}V + \frac{3}{2}\ D_x^\beta V \cdot D_x^{2\beta}V + \frac{3}{4}D_y^{2\beta}V + D_t^\beta\left(D_x^\beta V\right) = 0 \qquad 0 < \beta \leq 1, \quad t > 0 \tag{20}$$

$$let V(x,t,y) = V(\zeta), \qquad \zeta = \frac{\ell x^\beta}{\Gamma(1+\beta)} + \frac{\lambda t^\beta}{\Gamma(1+\beta)} + \frac{\mu y^\beta}{\Gamma(1+\beta)} \tag{21}$$

Where ℓ, λ and μ are constants then eq. (13) tansforms to

$$\frac{1}{4}\lambda^4 V^{''''} + \frac{3}{2}\ell^3 V^{'}\ V^{''} + \frac{3}{4}\mu^2 V^{''} + \ell\ \lambda V^{''} = 0$$

Integrate this equation, we obtain

$$\lambda^4 V^{'''} + 3\ \ell^3 V^{'2} + (3\mu^2 + 4\ell\ \lambda)\ V^{'} = 0 \tag{22}$$

By balancing the nonlinear term and the highest order derivative, we obtain $n = 1$, so eq. (15)has the solution

$$V(\zeta) = C_0 + C_1\psi \tag{23}$$

And the derivatives of V are

$$V^{'}(\zeta) = C_1\sigma + C_1\psi^2$$

$$V^{''}(\zeta) = 2C_1\sigma\psi + 2C_1\psi^3$$

$$V^{'''}(\zeta) = 2C_1\sigma^2 + 8C_1\sigma\psi^2 + 6C_1\psi^4$$

Substitute these derivatives in eq. (15)and equate the coefficients of ψ^4 , ψ^2, ψ^0

$$\psi^4:\ 6C_1\lambda^4 + 3\ell^3C_1^2 = 0$$

$$\psi^2:\ 8C_1\ \sigma\ \lambda^4 + 6C_1^2\ \ell^3\sigma + C_1(3\mu^2 + 4\ell\ \lambda) = 0$$

$$\psi^0:\ 2C_1\ \sigma^2\ \lambda^4 + 3\ell^3C_1^2 + C_1\sigma(3\mu^2 + 4\ell\ \lambda) = 0$$

We solve this system by using Mathematica Program, we obtain

$$C_1 = \pm\frac{\sqrt{2}}{3^{\frac{5}{4}}\mu^{\frac{5}{2}}}, \qquad \ell = \mp\frac{3^{\frac{3}{4}}\mu^{\frac{3}{2}}}{\sqrt{2}}, \qquad \lambda = \pm\frac{3^{\frac{1}{4}}\sqrt{\mu}}{\sqrt{2}},\ \sigma = -1 \tag{24}$$

So the solutions of eq. (13) by using (12,14,16,17) are:

$$V_1(x,t,y) = C_0 \mp \frac{\sqrt{2}}{3^{\frac{5}{4}}\mu^{\frac{5}{2}}}\tanh\left(\mp\frac{3^{\frac{3}{4}}\mu^{\frac{3}{2}}x^\beta}{\sqrt{2}\Gamma(1+\beta)} + \frac{3^{\frac{1}{4}}\sqrt{\mu}t^\beta}{\sqrt{2}\Gamma(1+\beta)} + \frac{\mu y^\beta}{\Gamma(1+\beta)}\right) \tag{25}$$

$$V_2(x,t,y) = C_0 \pm \frac{\sqrt{2}}{3^{\frac{5}{4}}\mu^{\frac{5}{2}}}\coth\left(\mp\frac{3^{\frac{3}{4}}\mu^{\frac{3}{2}}x^\beta}{\sqrt{2}\Gamma(1+\beta)} + \frac{3^{\frac{1}{4}}\sqrt{\mu}t^\beta}{\sqrt{2}\Gamma(1+\beta)} + \frac{\mu y^\beta}{\Gamma(1+\beta)}\right) \tag{26}$$

5. Conclusion

We used the sub-equation method to solve two non linear fractional partial differential equations, namely,namely the space time fractional potential Kadomstev–Petviashvili (PKP) equation and Burgers' equation. This method is based on the balancing principle, so it can be applied to others fractional partial differential equations which satisfy this principle. By using this method, we obtain more general exact solutions of many applications of non linear fractional partial differential equations.

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