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Research Article

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New Methods to Construct Rays Refracted by Flat and Spherical Surfaces

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Abstract The direction taken by a light ray reaching the interface between two transparent media of refractive indexes n_1 and n_2, is determined by the Snell-Descartes sine law $(n_1 \sin i_1 = n_2 \sin i_2)$. In practice, whatever the shape of the surface, the use of this formula is not easy, during teaching session, because measuring and drawing angles on a blackboard or on a sheet of paper is very challenging even with the help of a protractor. In the case of flat surfaces, the geometrical constructions of Snell-Descartes and Huygens are used and in the case of spherical surfaces, out of Gauss conditions, the Young construction is adopted. In this study two innovative methods, not based on the use of arcs of circles as in the three constructions mentioned above, are proposed. The first method will simplify constructions for flat surfaces and the second do the same for spherical surfaces, out of Gauss conditions.

Keywords Ray optics, flat surface, spherical surface, teaching, pedagogy

Introduction

Drawing circles and measuring angles on a blackboard or on a sheet without the use of a compass and a protractor is not an easy task very for teachers and students that deal with geometry. In the case of ray optics, these kinds of representations are very common, especially when it comes to drawing rays refracted by flat or curved surfaces.

In the case of a flat refracting surface, two methods are generally proposed: the so-called Snell-Descartes construction [1, 2] which uses two circles with radii equal to the refractive indexes of the two media and the construction of Huygens [3] which uses two circles with radii equal to the inverse of the refractive indexes of the two media. On the other hand, in the case of spherical refracting surfaces, out of the Gauss conditions [4], one can apply the Young construction that uses two specific arcs of circles [5, 6]. Under Gauss conditions, some rules exist that make the constructions very easy via the focal points and planes.

In this study we remain in the context of ray optics and deal with non-paraxial rays propagating in homogeneous and transparent media. The wave aspect is neglected, and we study single rays traveling through media of different refractive indexes.

For flat refracting surfaces, we describe the method of constructing refracted rays without the help of circles and angles measurement. We first measure the position of a given point, A, located somewhere on the line that holds the ray of incidence, calculate the position of a pseudo point-image, A′, belonging to the non-point image of A and finally draw the refracted ray. The calculation of the position of A′ is done using a formula that will be demonstrated in the next sections.

For spherical surfaces, we propose a way of drawing the refracted rays by calculating the position of a point, K', derived from that of a special point K. The special point, K, is located at the intersection of the ray of incidence and the axis passing by C and perpendicular to the line joining C and I. C is the center of the spherical refracting surface and I the point of incidence. To achieve that, we use another formula that will also be demonstrated in the next sections.

Plane refracting surface

To perform the Snell-Descartes construction [1, 2, 9] on a flat surface separating two transparent and homogeneous media 1 and 2 with refractive indexes n_1 and n_2, respectively, one can follow these steps:

- in the medium with refractive index n_2 (the medium where the refraction occurs), draw two concentric half circles of respective radii n_1 and n_2 centered at the point of incidence I;
- draw a line perpendicular to the surface and passing by the point N as shown in Figure 1. N is the intersection of circle of radius n_1 and the extension of the incident ray. The extension of the incident ray intersects also the circle of radius n_2 at point C.
- The refracted ray starts at the point of incidence I and passes through the point C.

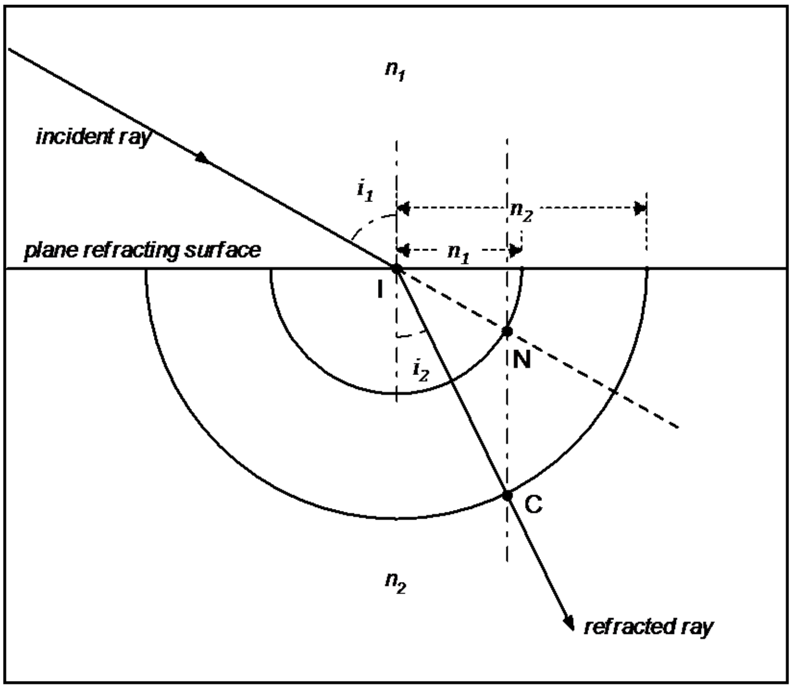

Figure 1: Construction of a refracted ray by the use of Snell-Descartes method on a flat surface separating two transparent and homogeneous media. $\boldsymbol{n_1}$ is the refractive index of the medium above the surface and $\boldsymbol{n_2}$ is the refractive index of the medium below the same surface.

Concerning the Huygens construction [3, 7, 8], that uses the inverse of the refractive indexes, the procedure is as follows:

- in the medium with refractive index n_2, draw two concentric half circles of respective radii $1/n_1$ and $1/n_2$ centered at the point of incidence I. The extension of the ray of incidence intersects the circle of radius $1/n_1$ at point N as shown in Figure 2.
- draw the line passing by N and perpendicular to the extension of the incident ray. This line intersects the surface at point T;
- draw the line that passes by T and tangent to the circle of radius $1/n_2$ at point C.The refracted ray starts from the point of incidence I and passes through point C.

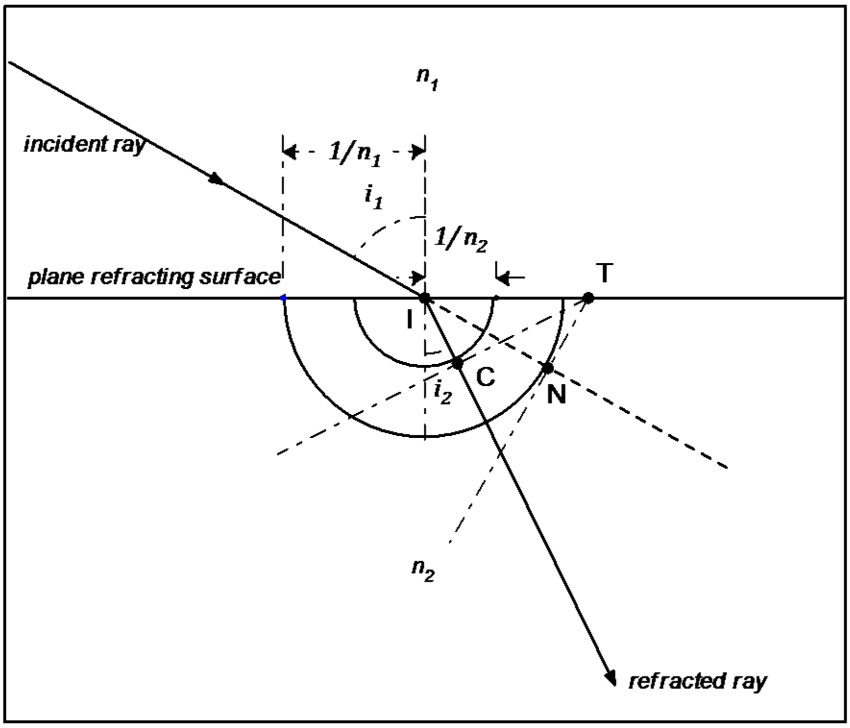

Figure 2: Construction of a refracted ray using Huygens method on a flat surface separating two transparent and homogeneous media.

Most teachers and students that use Snell-Descartes or Huygens constructions find it difficult to draw circles and measure accurately angles on a blackboard. That's why it may be interesting to hold out a way of representing refracted rays that don't use circles or involve angle measurements.

The method proposed is about drawing refracted rays, in the case of flat surfaces, and consists in the calculation of the position of a special point A′ (called "pseudopoint-image"). A′ is an element of the set of point-images $\{A'_i\}$ of a given point-object A that is located on the ray of incidence. Out of the Gauss conditions [4, 10, 11], a point-object does not give a point-image through a plane refracting surface, it rather gives a set of scattered point-images $\{A'_i\}$ which corresponds to a fuzzy image. The position of A′ is given by the equation (1) derived from the classic plane refracting surface equation:

$$\overline{HA'} = \overline{HA}\frac{n_2}{n_1}\sqrt{1+\left(\frac{\overline{HI}}{\overline{HA}}\right)^2\left(1-\left(\frac{n_1}{n_2}\right)^2\right)} \tag{1}$$

H is the orthogonal projection of A on the plane refracting surface.

The position of the pseudopoint-image

The term "pseudopoint-image" refers to the fact that a point-object's image is not a point-image because a plane refracting surface is not stigmatic.

When the conditions of stigmatism are satisfied, all the rays coming from a point object A have their corresponding refracted rays converging on a single point (the point where the point-image is formed). Out of the conditions of stigmatism, the refracted rays do not converge on one point. Consider the ray $(\mathcal{R}_P)$ coming from A and hitting the surface perpendicularly, its corresponding refracted ray $(\mathcal{R}'_P)$ is not deviated, it remains parallel to the incident ray along the same line. Any other non-perpendicular ray $(\mathcal{R}_i)$ coming from A and striking the surface has its refracted ray $(\mathcal{R}'_i)$ intersecting $(\mathcal{R}'_P)$ at a point A'_i. A'_i is what we call a pseudo "point-image". For any non-perpendicular incident ray from A, it corresponds a pseudo "point-image" A'_i formed at the intersection of its corresponding refracted ray and the line $(\mathcal{R}'_P)$. The set of A'_i are aligned along the line $(\mathcal{R}_P)$ and are part of the blurred image of A by the diopter.

In order to determine the position of A′ without measuring the angles of incidence and refraction, let's start from the following classic plane refracting surface equation [12] and eliminate the angles:

$$\overline{HA'} = \overline{HA}\frac{n_2}{n_1}\sqrt{\frac{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2 i_1}{1-\sin^2 i_1}} \tag{2}$$

Considering the following trigonometric equation:

$$\sin^2 \alpha = \frac{\tan^2 \alpha}{1+\tan^2 \alpha}$$

If we take $\overline{HA} = y_A$ and $\overline{HI} = x_I$, then we have:

$$\sin^2 i_1 = \frac{(x_I/y_A)^2}{1+(x_I/y_A)^2} \tag{3}$$

That is:

$$\sin^2 i_1 = \frac{{x_I}^2}{{x_I}^2 + {y_A}^2} \tag{4}$$

Replacing in equation (2) we obtain:

$$y_{A'} = y_A\frac{n_2}{n_1}\sqrt{1+\left(\frac{x_I}{y_A}\right)^2\left(1-\left(\frac{n_1}{n_2}\right)^2\right)} \tag{5}$$

That leads to:

$$\overline{HA'} = \overline{HA}\frac{n_2}{n_1}\sqrt{1+\left(\frac{\overline{HI}}{\overline{HA}}\right)^2\left(1-\left(\frac{n_1}{n_2}\right)^2\right)} \tag{6}$$

There is no angular parameter in this equation. H is taken as the origin of the orthogonal frame of reference $\left(\mathrm{H}, \overrightarrow{HA}, \overrightarrow{HI}\right)$. $\overline{HA}$ and $\overline{HI}$(y_A and x_I) can be measured using a ruler and thereafter $\overline{HA'}$ is directly calculated.

Drawing the Refracted Ray

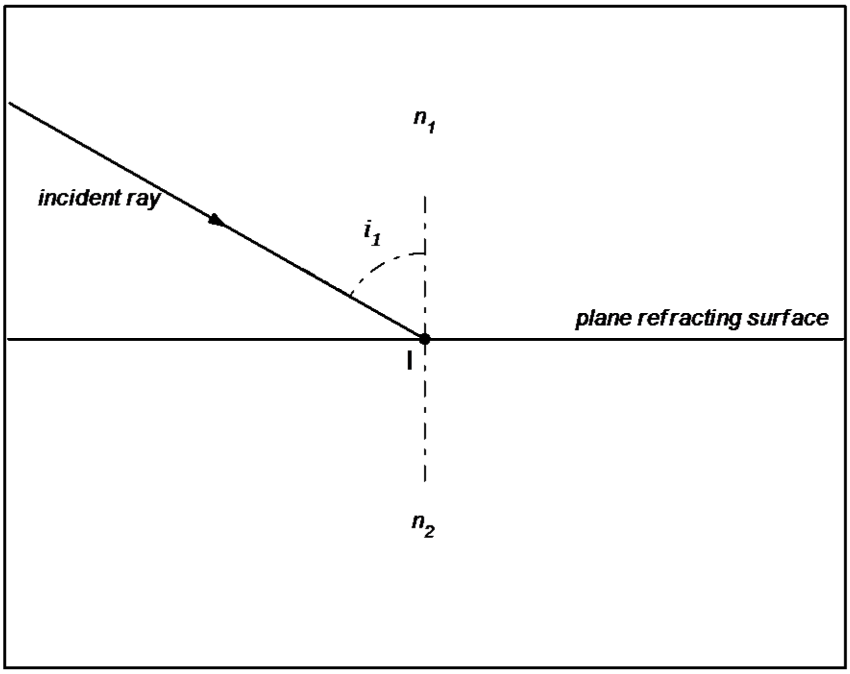

Figure 3: Ray of incidence hitting a flat refracting surface at an angle $\boldsymbol{i_1}$.

To determine the refracted ray direction, one can adopt the following steps:

- choose any point A on the incident ray, its orthogonal projection on the interface H is taken as the origin of reference. The point A can be chosen anywhere on the ray of incidence, one must make a choice that facilitate measurements with a ruler (see Figure 4). With $\overline{HA} = y_A$ and $\overline{HI} = x_I$;

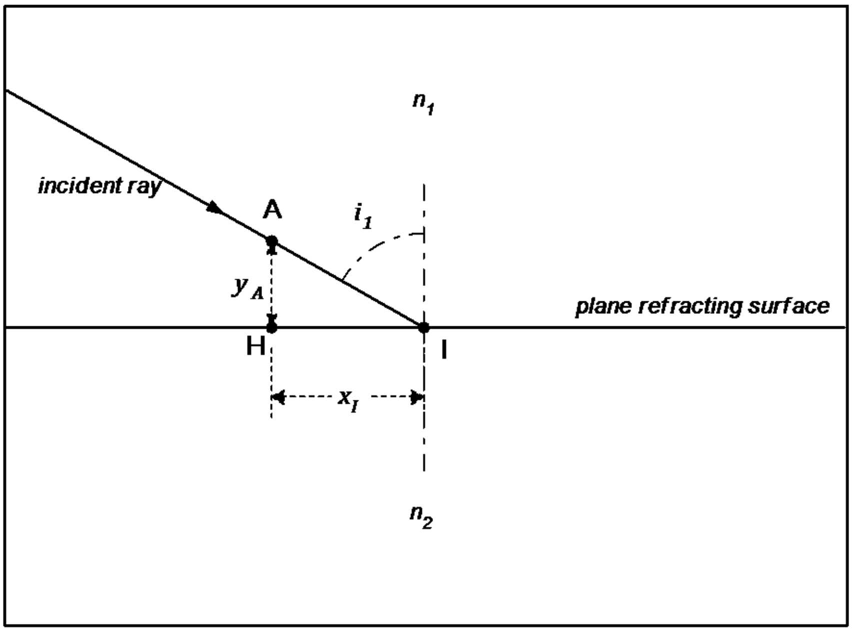

Figure 4: An arbitrary choice of a point-object A, somewhere on the incident ray, and its projection H on the refracting surface.

- calculate the value of $\overline{HA'}$ using equation (1, 5 or 6) ($\overline{HA}$ is measured, n_1 and n_2 are known);
- place the point A′ from Hon the graph.A′ is located on the line passing by H and perpendicular to the flat interface (see Figure 5).With $\overline{HA'} = y_{A'}$;

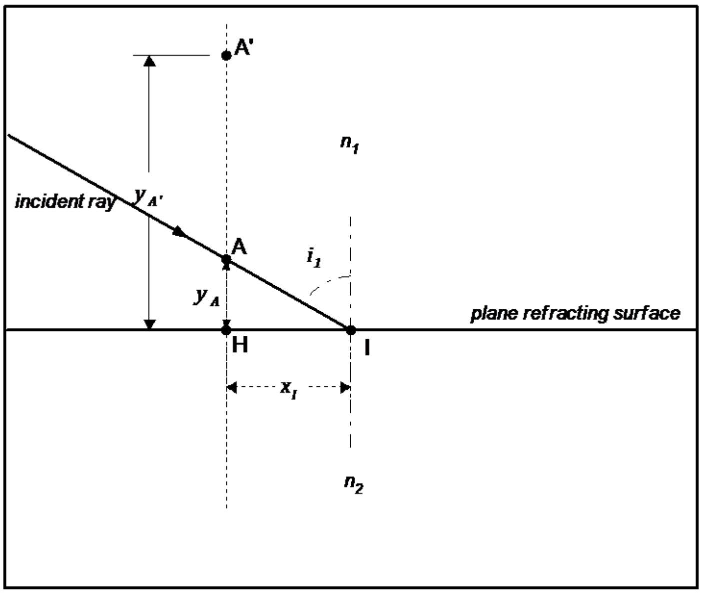

Figure 5: Point-object A and its calculated corresponding "pseudopoint-image" A′.

- The refracted ray is on the line $\overrightarrow{A'I}$, it starts on the point of incidence I and continues inside the second medium with n_2 refractive index (Figure 6).

Thus, the use of equation (1) makes it possible to draw easily a light ray refracted by a plane refracting surface compared to the constructions of Snell-Descartes and Huygens that use a couple of circles.

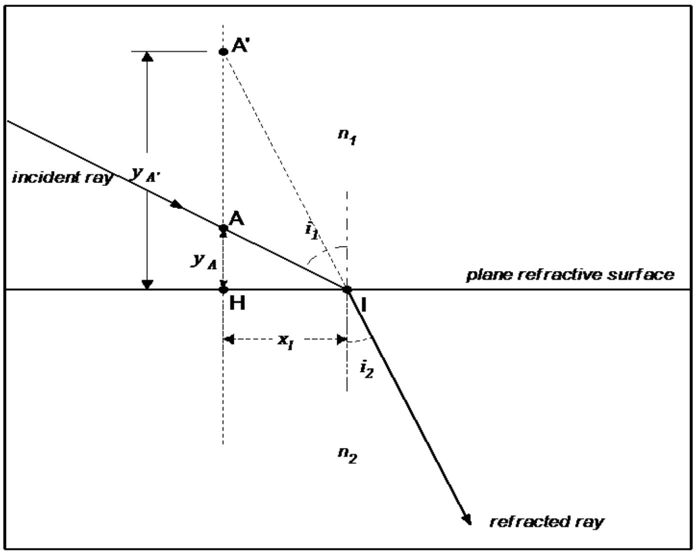

Figure 6: The refracted ray is the extension of the line $\overrightarrow{A'I}$ in the medium where the refraction occurs

In addition, one can calculate the incidence and refraction angles using the following formulas:

$$\tan i_1 = \frac{|x_I|}{|y_A|} \tag{7}$$

and

$$\tan i_2 = \frac{|x_I|}{|y_{A'}|} \tag{8}$$

Spherical refracting surface

To perform the Young construction [5, 6] on a spherical surface separating two transparent and homogeneous media 1 and 2 with refractive indexes n_1 and n_2, respectively, one can follow these steps:

- in the medium with refractive index n_1, draw two concentric arcs of circle of respective radii $\frac{n_1}{n_2}R$ and $\frac{n_2}{n_1}R$ centered at the center of curvature C of the refracting surface (Figure 7). R is the radius of the refracting surface and let N be the intersection between the incident ray and the arc of radius $\frac{n_2}{n_1}R$;
- draw a line passing by the N and C.. This line intersects the arc of radius $\frac{n_1}{n_2}R$ at point L.
- The refracted ray starts on the point of incidence Iand follow the direction $\overrightarrow{LI}$.

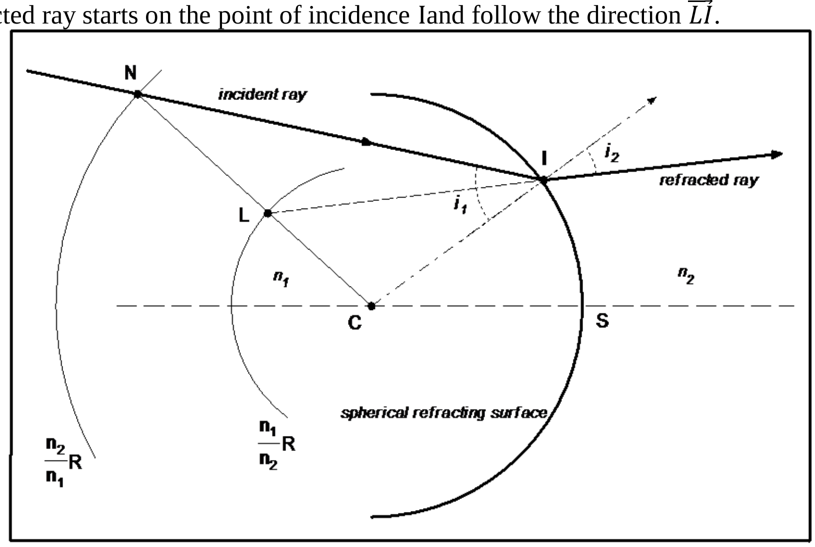

Figure 7: Construction of a refracted ray using Young method on a spherical surface separating two transparent and homogeneous media

To avoid the drawing arcs, we propose a method based on the calculation of the position of a point K′(called "point support of the refracted ray") derived from the position of a special point K. The point K is located at the intersection of the ray of incidence and the axis perpendicular to the line joining the center of the spherical surface C and the point of incidence I. To achieve that, the following formula, that will be demonstrated in the next session, is used:

$$\overline{CK'} = \pm \frac{1}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 \left(\frac{1}{R^2} + \frac{1}{\overline{CK}^2}\right) - \frac{1}{R^2}}} \tag{9}$$

Position of the Point Support of the Refracted Ray

Consider the orthogonal frame of reference $\left(\mathrm{C}, \overrightarrow{CI}, \overrightarrow{CK}\right)$. Points K and K′ are the intersections of the line perpendicular to $\overrightarrow{CI}$ passing by centre of curvature C and the incident and refracted rays respectively (Figure 11 with $\alpha \equiv \mathrm{angle(KC, KI)}$, $\gamma \equiv \mathrm{angle(CK, CI)} = \frac{\pi}{2}$, $\beta \equiv \mathrm{angle(K'C, K'I)}$).

- Triangle $\widehat{KCI}$:

$$\frac{\overline{CK}}{\sin i_1} = \frac{\overline{KI}}{\sin \gamma} = \frac{\overline{CI}}{\sin \alpha} \Longrightarrow \sin i_1 = \frac{\overline{CK}}{\overline{KI}} \sin \gamma \Longrightarrow \sin i_1 = \frac{\overline{CK}}{\overline{KI}} \quad - \tag{10}$$

- Triangle $\widehat{K'CI}$:

$$\frac{\overline{CK'}}{\sin i_2} = \frac{\overline{K'I}}{\sin \gamma} = \frac{\overline{CI}}{\sin \beta} \Longrightarrow \sin i_2 = \frac{\overline{CK'}}{\overline{K'I}} \sin \gamma \Longrightarrow \sin i_1 = \frac{\overline{CK'}}{\overline{K'I}} \quad - \tag{11}$$

- Using Snell-Descartes law $n_1 \sin i_1 = n_2 \sin i_2$:

$$n_1 \frac{\overline{CK}}{\overline{KI}} = n_2 \frac{\overline{CK'}}{\overline{K'I}} \Longrightarrow n_1 \overline{CK} \cdot \overline{K'I} = n_2 \overline{CK'} \cdot \overline{KI} \quad - \tag{12}$$

- $\widehat{KCI}$ and $\widehat{K'CI}$ are rectangle triangles:

$$n_1{}^2 \overline{CK}^2 \left(\overline{CK'}^2 + \overline{CI}^2\right) = n_2{}^2 \overline{CK'}^2 \left(\overline{CK}^2 + \overline{CI}^2\right)$$

$$\overline{CK'}^2 + \overline{CI}^2 = \left(\frac{n_2}{n_1}\right)^2 \left(\frac{\overline{CK}^2 + \overline{CI}^2}{\overline{CK}^2}\right) \overline{CK'}^2$$

$$\left(\left(\frac{n_2}{n_1}\right)^2 \left(\frac{\overline{CK}^2 + \overline{CI}^2}{\overline{CK}^2}\right) - 1\right) \overline{CK'}^2 = \overline{CI}^2$$

$$\overline{CK'} = \pm \frac{\overline{CI}}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 \left(\frac{\overline{CK}^2 + \overline{CI}^2}{\overline{CK}^2}\right) - 1}}$$

$$\overline{CK'} = \pm \frac{1}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 \left(\frac{1}{\overline{CI}^2} + \frac{1}{\overline{CK}^2}\right) - \frac{1}{\overline{CI}^2}}}$$

Since $CI = R$ (the radius of the spherical surface):

$$\overline{CK'} = \pm \frac{1}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 \left(\frac{1}{R^2} + \frac{1}{\overline{CK}^2}\right) - \frac{1}{R^2}}} \tag{13}$$

$\overline{CK}$ and R can be measured using a ruler and thereafter $\overline{CK'}$ is calculated.

The minus sign in equation (13) leads to a wrong solution because the ray of incidence must not be on the same side as the ray of refraction with respect to the normal CI. The refracted ray must cross the normal.

Determination of the Refracted Ray

The equation (13) allows the determination of the refracted ray once $\overline{CK'}$ calculated.

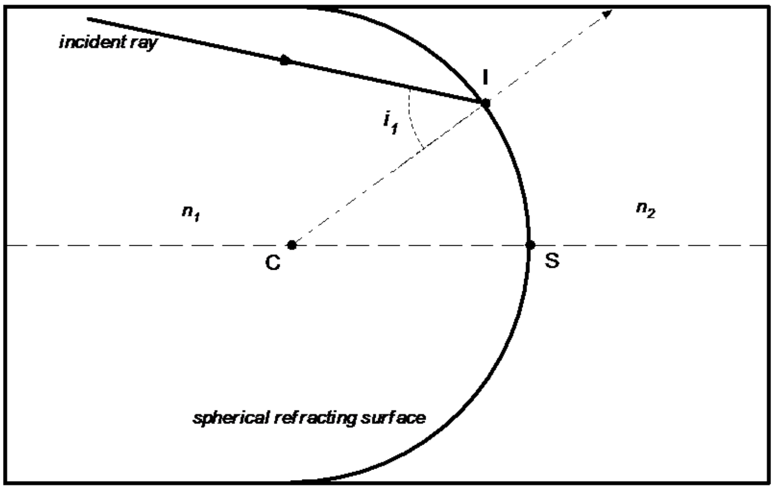

Figure 8: Ray of incidence hitting a spherical refracting surface at the point of incidence I

To determine the refracted ray direction, these steps can be followed:

- place on the figure the point K, intersection of the ray of incidence and the line perpendicular to $\overrightarrow{CI}$ and measure the length CK using a ruler for example (Figure 9);

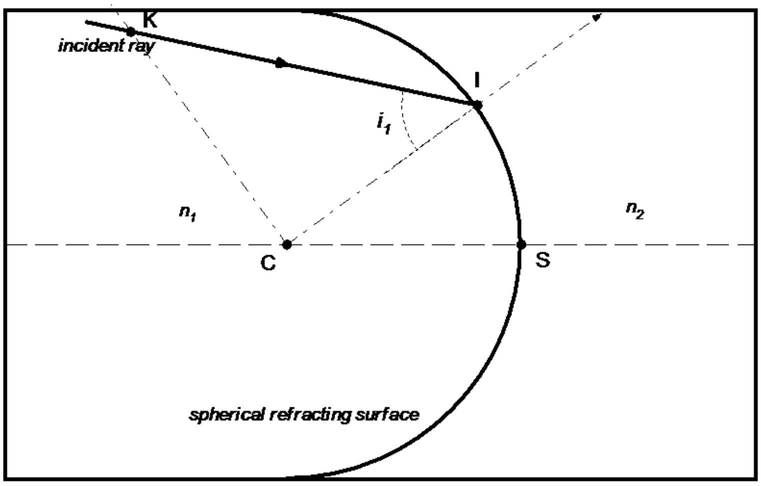

*Figure 9: Positioning the point **K**, intersection of the ray of incidence and the line perpendicular to the line passing by the center C and the point of incidence I*

- calculate the position of the point K′ using equation (13) and place it (Figure 10);

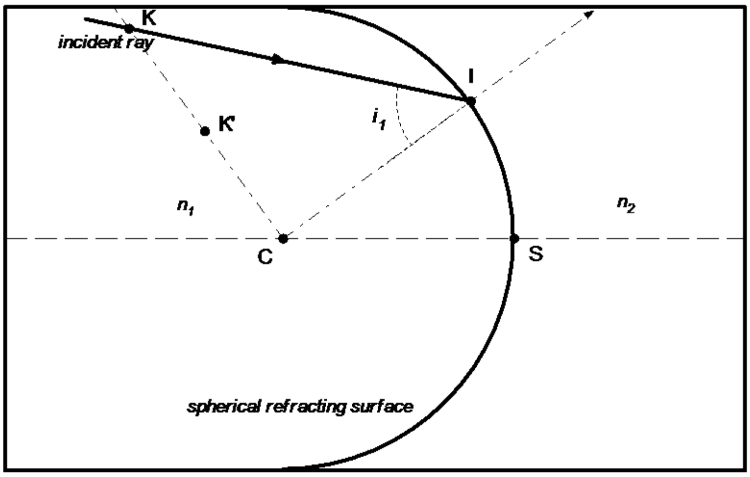

*Figure 10: Positioning the point **K′** by measuring the length CI and using the equation for the calculation*

- the refracted ray is on the line $\overrightarrow{K'I}$, it starts from the point of incidence I and continues inside the second medium (Figure 11).

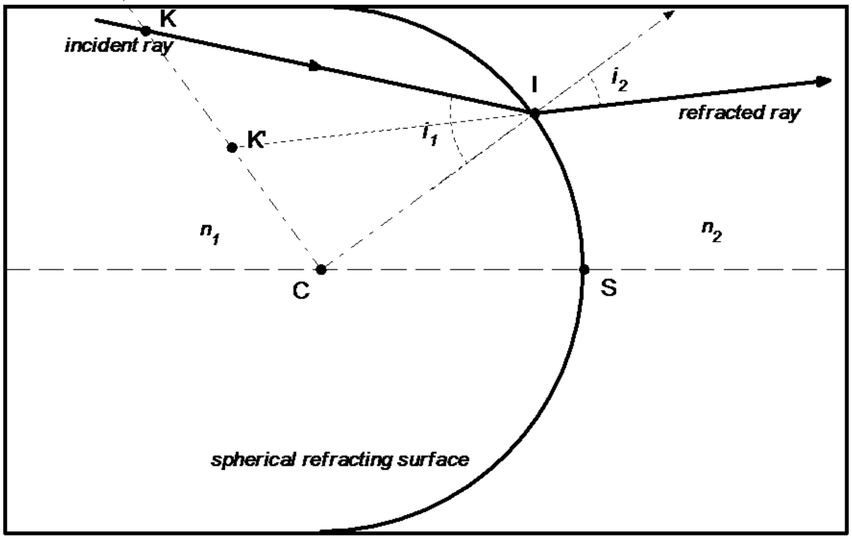

Figure 11: The refracted ray is the extension of the line $\boldsymbol{K'I}$ *in the medium where the refraction occurs*

The incidence and refraction angles can be deduced using the following formulas:

$$\tan i_1 = \frac{|\overline{CK}|}{R} \tag{14}$$

and

$$\tan i_2 = \frac{|\overline{CK'}|}{R} \tag{15}$$

Conclusion

In this paper, we proposed two easy methods to construct rays refracted by flat and spherical surfaces respectively (only a ruler is needed) that can help teachers and students to avoid the drawings of circles needed in Snell-Descartes, Huygens and Young methods of construction and to avoid angle measurements when Snell-Descartes sine law is directly used.

For flat refracting surfaces, the method consists in the calculation of the position of a so-called pseudopoint-image of a point-object chosen arbitrarily on the ray of incidence. The position of the pseudopoint-image is calculated using an equation that depends on the refractive indexes of the two media and on the position of the point-object. The line that passes by the positions of the pseudopoint-image and the point of incidence determines the direction of the refracted ray.

When it comes to deal with spherical refracting surfaces, the positions of two special points (K and K′) are determined using another equation that depends on the indexes of refraction of the two media, the radius of the refracting surface and the position of the special point K. This second equation allows to construct easily the refracted rays out of Gauss conditions.

These methods can interest people working on ray tracing on plane or spherical refracting surfaces and the equations can be implemented in the ray-tracing numerical programs.

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