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Some Fractional Operators of the Analytic Functions with Negative Coefficients

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Abstract In this paper, we introduce and investigate two new subclasses $S^*C(\alpha,\beta;\gamma)$ and $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$ of analytic functions in the open unit disk. The object of the present paper is to derive characteristic properties of these classes. Also, several coefficient bound estimates for the functions belonging to these classes are also given. Distortion theorems for the functions belonging to the class $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$ are also proved.

Keywords Analytic function, coefficient bound, starlike function, convex function
AMS Subject Classification: 30C45, 30C50, 30C55

1. Introduction and preliminaries

In this section, we give the necessary information and preliminaries which shall need in our investigation.

Let A be the class of analytic functions $f(z)$ in the open unit disk $U=\{z\in\mathbb{C}:\ |z|<1\}$, normalized by $f(0)=0=f'(0)-1$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}. \tag{1.1}$$

It is well-known that an analytic function $f:\mathbb{C}\to\mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1=z_2$ if $f(z_1)=f(z_2)$ or $f(z_1)\neq f(z_2)$ if $z_1\neq z_2$. We denote by S the subclass of A consisting of functions which are also univalent in U.

Let T denote the subclass of all functions $f(z)$ in A of the form

$$f(z)=z-a_2z^2-a_3z^3-\cdots-a_nz^n-\cdots=z-\sum_{n=2}^{\infty}a_nz^n,\ a_n\geq 0. \tag{1.2}$$

We will also denote by $S^*(\alpha)$ and $C(\alpha)$ the subclasses of S that are, respectively, starlike and convex of order α $(\alpha\in[0,1))$ in the open unit disk U. By definition, we have (see for details, [5, 6], also [15])

$$S^*(\alpha)=\left\{f\in A:\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in U\right\},\ \alpha\in[0,1),$$

and

$$C(\alpha)=\left\{f\in A:\ \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\ z\in U\right\},\ \alpha\in[0,1).$$

Let's $TS^*(\alpha)=S^*(\alpha)\cap T$ and $TC(\alpha)=C(\alpha)\cap T$.

Interesting generalization of the functions classes $S^*(\alpha)$ and $C(\alpha)$, are classes $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$, which defined by

$$S^*(\alpha,\beta)=\left\{f\in A:\ \mathrm{Re}\left(\frac{zf'(z)}{\beta zf'(z)+(1-\beta)f(z)}\right)>\alpha,\ z\in U\right\},\ \alpha,\beta\in[0,1)$$

and

$$C(\alpha,\beta)=\left\{f\in A:\ \mathrm{Re}\left(\frac{f'(z)+zf''(z)}{f'(z)+\beta zf''(z)}\right)>\alpha,\ z\in U\right\},\ \alpha,\beta\in[0,1),$$

respectively.

We will denote $TS^*(\alpha,\beta)=S^*(\alpha,\beta)\cap T$ and $TC(\alpha,\beta)=C(\alpha,\beta)\cap T$. The classes $TS^*(\alpha,\beta)$ and $TC(\alpha,\beta)$ were extensively studied by Altintaş and Owa [3] and certain conditions for hypergeometric functions and generalized Bessel functions for these classes were studied Mosutafa [8] and Porwal and Dixit [12].

Inspired by the works mentioned above, we introduce a unification of the functions classes $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$ defined as follows

Definition 1.1. *A function $f\in A$ given by (1.1) is said to be in the class $S^*C(\alpha,\beta;\gamma),\ \alpha\ge 0,\ \beta\ge 0,\ \gamma\in[0,1]$ if the following condition is satisfied*

$$\mathrm{Re}\left(\frac{zf'(z)+\gamma z^2f''(z)}{\gamma z\left(f'(z)+\beta zf''(z)\right)+(1-\gamma)\left(\beta zf'(z)+(1-\beta)f(z)\right)}\right)>\alpha, z\in U.$$

Note that for $\gamma=0$ and $\gamma=1$, $S^*C(\alpha,\beta;0)=S^*(\alpha,\beta),\ \alpha\ge 0,\ \beta\ge 0$ and $S^*C(\alpha,\beta;1)=C(\alpha,\beta),\ \alpha\ge 0,\ \beta\ge 0$, respectively. We will also denote $TS^*C(\alpha,\beta;\gamma)=S^*C(\alpha,\beta;\gamma)\cap T$

For the custom values of the parameters, the above-defined classes include several simple subclasses. Here are some of these special cases as follows:

1) For $\gamma=0$, we get the subclass $TS^*C(\alpha,\beta;0)=TS^*(\alpha,\beta),\ \alpha,\beta\in[0,1)$ consisting of the functions $f\in T$ satisfying the following condition

$$\mathrm{Re}\left(\frac{zf'(z)}{\beta zf'(z)+(1-\beta)f(z)}\right)>\alpha,\ z\in U,$$

which was studied by Altıntaş (see [1,2 and 4]);

2) For $\beta=0$, $\gamma=0$, we obtain the classes $S^*C(\alpha,0;0)=S*(\alpha)$, $TS^*C(\alpha,0;0)=TS^*(\alpha)$, $\alpha\in[0,1)$. These classes are well-known starlike functions of order α and was studied by several researchers (see for example Silverman [13]);

3) For $\beta=0$, $\gamma=1$, we get the classes $S^*C(\alpha,0;1)=C(\alpha)$, $TS^*C(\alpha,0;1)=TC(\alpha)$, $\alpha\in[0,1)$. These classes are well-known convex functions of order α and were studied by several researchers (see for example Silverman [13]);

4) For $\gamma=1$, we get the class $TS^*C(\alpha,\beta;1)=TC(\alpha,\beta)$, $\alpha,\beta\in[0,1)$ consisting of the functions $f\in T$ satisfying the following condition

$$\operatorname{Re}\left(\frac{f'(z)+zf''(z)}{f'(z)+\beta zf''(z)}\right)>\alpha, z\in U,$$

which was studied by Altıntaş [3];

5) For $\beta=0$, we obtain the classes $S^*C(\alpha,0;\gamma)=S^*C(\alpha,\gamma;1)$, $TS^*C(\alpha,0;\gamma)=TS^*C(\alpha,\gamma;1)$, $\alpha\in[0,1)$, $\gamma\in[0,1]$ consisting of the functions $f\in A$ satisfying the following condition

$$\operatorname{Re}\left(\frac{zf'(z)+\gamma z^2 f''(z)}{\gamma zf'(z)+(1-\gamma)f(z)}\right)>\alpha, z\in U,$$

which was studied by Mustafa [9, when $\tau=1$] and the references cited in of them.

The object of the present paper is to examine characteristic properties of the classes $S^*C(\alpha,\beta;\gamma)$ and $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$. Coefficient bounds for the functions belonging to these classes are also determined. We also prove several distortion theorems involving certain operators of fractional calculus for the functions in the class $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$.

2. Coefficient bounds for the classes $S^*C(\alpha,\beta;\gamma)$ and $TS^*C(\alpha,\beta;\gamma)$

In this section, we will examine some characteristic properties of the subclasses $S^*C(\alpha,\beta;\gamma)$ and $TS^*C(\alpha,\beta;\gamma)$ of analytic functions in the open unit disk. Also, we give coefficient estimates for the functions belonging to these subclasses.

A sufficient condition for the functions in the class $S^*C(\alpha,\beta;\gamma)$ is given by the following theorem.

Theorem 2.1. *Let $f\in A$. Then, the function $f(z)$ belongs to the class $S^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)|a_n|\le 1-\alpha. \tag{2.1}$$

The result is sharp for the functions

$$f_n(z)=z+\frac{1-\alpha}{(1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)}z^n,\ z\in U,\ n=2,3,\ldots. \tag{2.2}$$

Proof. From the Definition 1.1, a function $f\in S^*C(\alpha,\beta;\gamma),\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)+\gamma z^2 f''(z)}{\gamma z(f'(z)+\beta zf''(z))+(1-\gamma)(\beta zf'(z)+(1-\beta)f(z))}\right)>\alpha. \tag{2.3}$$

We can easily show that condition (2.3) holds true if

$$\left|\frac{zf'(z)+\gamma z^2 f''(z)}{\gamma z\left(f'(z)+\beta zf''(z)\right)+(1-\gamma)\left(\beta zf'(z)+(1-\beta)f(z)\right)}-1\right|\leq 1-\alpha. \qquad (2.4)$$

Now, let us show that this condition is satisfied under hypothesis (2.1) of the theorem.

If we take into account the expansion series (1.1) of the function $f(z)$, we can write

$$\left|\frac{zf'(z)+\gamma z^2 f''(z)}{\gamma z\left(f'(z)+\beta zf''(z)\right)+(1-\gamma)\left(\beta zf'(z)+(1-\beta)f(z)\right)}-1\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-1)(1-\beta)a_n z^n}{z+\sum_{n=2}^{\infty}(1+(n-1)\gamma)(1+(n-1)\beta)a_n z^n}\right|\leq\frac{\sum_{n=2}^{\infty}(1+(n-1))\gamma)(n-1)(1-\beta))|a_n|}{1-\sum_{n=2}^{\infty}(1+(n-1)\gamma)(1+(n-1)\beta)|a_n|}.$$

As you can see the inequality (2.4) holds true if

$$\frac{\sum_{n=2}^{\infty}(1+(n-1))\gamma)(n-1)(1-\beta))|a_n|}{1-\sum_{n=2}^{\infty}(1+(n-1)\gamma)(1+(n-1)\beta)|a_n|}\leq 1-\alpha,$$

which is equivalent to

$$\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-1)(1-\beta)|a_n|\leq(1-\alpha)\left\{1-\sum_{n=2}^{\infty}(1+(n-1)\gamma)(1+(n-1)\beta)|a_n|\right\}.$$

But, this is the same of the condition (2.1).

Moreover, we can easily see that the inequality (2.1) is sharp for the functions $f_n(z)$ given by (2.2).

Thus, the proof of Theorem 2.1 is completed.

By setting $\gamma=0$ and $\gamma=1$ in Theorem 2.1, we can readily deduce the following results, respectively.

Corollary 2.1. *The function $f(z)$ defined by (1.1) belongs to the class $S^*(\alpha,\beta)$, $\alpha,\beta\in[0,1)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}(n-\alpha-(n-1)\alpha\beta)|a_n|\leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z)=z+\frac{1-\alpha}{n-\alpha-(n-1)\alpha\beta}z^n,\ z\in U,\ n=2,3,\dots.$$

Corollary 2.2. *The function $f(z)$ defined by (1.1) belongs to the class $C(\alpha,\beta)$, $\alpha,\beta\in[0,1)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}n(n-\alpha-(n-1)\alpha\beta)|a_n|\leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z)=z+\frac{1-\alpha}{n(n-\alpha-(n-1)\alpha\beta)}z^n,\ z\in U,\ n=2,3,\dots.$$

By taking $\beta=0$ in Corollary 2.1 and 2.2, respectively, we have the following results.

Corollary 2.3. (see [13, p. 110, Theorem 1]) *The function* $f(z)$ *defined by (1.1) belongs to the class* $S^*(\alpha),\ \alpha\in[0,1)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leq 1-\alpha .$$

The result is sharp for the functions

$$f_n(z)=z+\frac{1-\alpha}{n-\alpha}z^n,\ z\in U,\ n=2,3,\ldots .$$

Corollary 2.4. (see [13, p. 110, Corollary of Theorem1]) *The function* $f(z)$ *defined by (1.1) belongs to the class* $C(\alpha),\ \alpha\in[0,1)$ *if the following condition is satisfied*

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\leq 1-\alpha .$$

The result is sharp for the functions

$$f_n(z)=z+\frac{1-\alpha}{n(n-\alpha)}z^n,\ z\in U,\ n=2,3,\ldots .$$

For the function in the class $TS^*C(\alpha,\beta;\gamma)$, the converse of Theorem 2.1 is also true.

Theorem 2.2. *Let* $f\in T$. *Then, the function* $f(z)$ *belongs to the class* $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1),\ \gamma\in[0,1]$ *if and only if*

$$\sum_{n=2}^{\infty}\big(1+(n-1)\gamma\big)\big(n-\alpha-(n-1)\alpha\beta\big)a_n\leq 1-\alpha . \tag{2.5}$$

The result is sharp for the functions

$$f_n(z)=z-\frac{1-\alpha}{\big(1+(n-1)\gamma\big)\big(n-\alpha-(n-1)\alpha\beta\big)}z^n,\ z\in U,\ n=2,3,\ldots . \tag{2.6}$$

Proof. The proof of the sufficiency of theorem can be proved similarly to the proof of Theorem 2.1. That's why; we will prove only the necessity of the theorem.

Assume that $f\in TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1),\ \gamma\in[0,1]$, which is equivalent to

$$\operatorname{Re}\left\{\frac{zf'(z)+\gamma z^2f''(z)}{\gamma z\big(f'(z)+\beta zf''(z)\big)+(1-\gamma)\big(\beta zf'(z)+(1-\beta)f(z)\big)}\right\}>\alpha,\ z\in U .$$

Then, by simple computation, we obtain

$$\operatorname{Re}\left\{\frac{zf'(z)+\gamma z^2f''(z)}{\gamma z\big(f'(z)+\beta zf''(z)\big)+(1-\gamma)\big(\beta zf'(z)+(1-\beta)f(z)\big)}\right\}$$

$$=\operatorname{Re}\left\{\frac{z-\sum_{n=2}^{\infty}n\big(1+(n-1)\gamma\big)a_nz^n}{z-\sum_{n=2}^{\infty}\big(1+(n-1)\gamma\big)\big(1+(n-1)\beta\big)a_nz^n}\right\}>\alpha.$$

The last expression in the brackets of the above inequality is real if choose z real. Hence, from the previous inequality letting $z\to 1$ through real values, we obtain

$$1-\sum_{n=2}^{\infty} n\left[1+\gamma(n-1)\right]a_n \geq \alpha\left\{1-\sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)\left[1+\beta(n-1)\right]a_n\right\}.$$

But, this is the same of the condition (2.5).

This is clear that the inequality (2.5) is sharp for the functions $f_n(z)$ given by (2.6).

Thus, the proof of Theorem 2.2 is completed.

By taking $\gamma=0$ and $\gamma=1$ in Theorem 2.2, we arrive at the following results, respectively.

Corollary 2.5. *The function* $f(z)$ *defined by (1.2) belongs to the class* $TS^*(\alpha,\beta)$, $\alpha,\beta\in[0,1)$ *if and only if*

$$\sum_{n=2}^{\infty}\left(n-\alpha-(n-1)\alpha\beta\right)a_n \leq 1-\alpha .$$

The result is sharp for the functions

$$f_n(z)=z-\frac{1-\alpha}{n-\alpha-(n-1)\alpha\beta}z^n,\ z\in U,\ n=2,3,\ldots .$$

Corollary 2.6. *The function* $f(z)$ *defined by (1.2) belongs to the class* $TC(\alpha,\beta)$, $\alpha,\beta\in[0,1)$ *if and only if*

$$\sum_{n=2}^{\infty} n\left(n-\alpha-(n-1)\alpha\beta\right)a_n \leq 1-\alpha .$$

The result is sharp for the functions

$$f_n(z)=z-\frac{1-\alpha}{n\left(n-\alpha-(n-1)\alpha\beta\right)}z^n,\ z\in U,\ n=2,3,\ldots .$$

Remark 2.1. The results obtained in Corollary 2.5 and 2.6 have been provided in [3].

By taking $\beta=0$ in Corollary 2.5 and 2.6, respectively, we have the following results.

Corollary 2.7. (see [13, p. 110, Theorem 2]) *The function* $f(z)$ *defined by (1.2) belongs to the class* $TS^*(\alpha)$, $\alpha\in[0,1)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n \leq 1-\alpha .$$

The result is sharp for the functions

$$f_n(z)=z-\frac{1-\alpha}{n-\alpha}z^n,\ z\in U,\ n=2,3,\ldots .$$

Corollary 2.8. (see [13, p. 111, Corollary 2]) *The function* $f(z)$ *defined by (1.2) belongs to the class* $TC(\alpha)$, $\alpha\in[0,1)$ *if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha .$$

The result is sharp for the functions

$$f_n(z)=z-\frac{1-\alpha}{n(n-\alpha)}z^n,\ z\in U,\ n=2,3,\ldots .$$

From the Theorem 2.2, we have the following result.

Corollary 2.9. *If* $f\in TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$, *then*

$$|a_n| \le \frac{1-\alpha}{\left(1+(n-1)\gamma\right)\left(n-\alpha-(n-1)\alpha\beta\right)},\ n=2,3,\dots.$$

Remark 2.2. Numerous consequences of Corollary 2.9 can indeed be deduced by specializing the various parameters involved. Many of these consequences were proved by earlier studies on the subject (cf., e.g., [1, 13, 16]).

From the Theorem 2.2, we can readily deduce the following result.

Theorem 2.3. *Let the function* $f(z)$ *definition by (1.2) belongs to the class* $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta \in [0,1)$, $\gamma \in [0,1]$ *. Then,*

$$\sum_{n=2}^{\infty} |a_n| \le \frac{1-\alpha}{(1+\gamma)\left(2-(1+\beta)\alpha\right)} \tag{2.7}$$

and

$$\sum_{n=2}^{\infty} n|a_n| \le \frac{2(1-\alpha)}{(1+\gamma)\left(2-(1+\beta)\alpha\right)}. \tag{2.8}$$

Proof. Assume that $f \in TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta \in [0,1)$, $\gamma \in [0,1]$. Then, it follows from (2.5) that

$$(1+\gamma)\left(2-(1+\beta)\alpha\right)\sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)\left(n-\alpha-(n-1)\alpha\beta\right)a_n \le 1-\alpha; \tag{2.9}$$

that is,

$$(1+\gamma)\left(2-(1+\beta)\alpha\right)\sum_{n=2}^{\infty} a_n \le 1-\alpha.$$

But, this is the same of the inequality (2.7).

On the other hand, similarly to (2.9), we can write

$$\left(2-(1+\beta)\alpha\right)\sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)a_n \le \sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)\left(n-\alpha-(n-1)\alpha\beta\right)a_n \le 1-\alpha.$$

So,

$$\left(2-(1+\beta)\alpha\right)\sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)a_n \le 1-\alpha.$$

The last inequality equivalent to

$$\left(2-(1+\beta)\alpha\right)\gamma\sum_{n=2}^{\infty} n a_n \le 1-\alpha+\left(2-(1+\beta)\alpha\right)(\gamma-1)\sum_{n=2}^{\infty} a_n.$$

Using the inequality (2.7) to the last inequality, we obtain

$$\left(2-(1+\beta)\alpha\right)\gamma\sum_{n=2}^{\infty} n a_n \le \frac{2\gamma(1-\alpha)}{1+\gamma}.$$

This completes the proof of the inequality (2.8).

Thus, the proof of Theorem 2.3 is completed.

By setting $\gamma = 0$ and $\gamma = 1$ in Theorem 2.3, we arrive at the following results, respectively.

Corollary 2.10. *Let the function* $f(z)$ *defined by (1.2) belongs to the class* $TS^*(\alpha,\beta)$, $\alpha,\beta \in [0,1)$. *Then,*

$$\sum_{n=2}^{\infty} |a_n| \le \frac{1-\alpha}{2-(1+\beta)\alpha} \quad \text{and} \quad \sum_{n=2}^{\infty} n|a_n| \le \frac{2(1-\alpha)}{2-(1+\beta)\alpha}.$$

Corollary 2.11. *Let the function* $f(z)$ *defined by (1.2) belongs to the class* $TC(\alpha,\beta)$, $\alpha,\beta\in[0,1)$. *Then,*

$$\sum_{n=2}^{\infty}|a_n|\le\frac{1-\alpha}{2\left(2-(1+\beta)\alpha\right)}\text{ and }\sum_{n=2}^{\infty}n|a_n|\le\frac{1-\alpha}{2-(1+\beta)\alpha}.$$

Remark 2.3. Numerous consequences of the coefficient inequalities obtained in the Theorems 2.1 and 2.2 and in the Corollaries 2.10 and 2.11 can indeed be deduced by specializing the various parameters involved.

3. Some properties of the function class $TS^*C(\alpha,\beta;\gamma)$

In this section, we will examine some interesting properties of the class $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$.

Theorem 3.1. *The subclass* $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1),\gamma\in[0,1]$ *of the analytic functions in the open unit disk is convex set.*

Proof. Assume that each of the functions $f,g\in TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1),\gamma\in[0,1]$ with

$$g(z)=z-\sum_{n=2}^{\infty}b_nz^n,\ b_n\ge 0. \tag{3.1}$$

Then, for $\lambda\in[0,1]$, we write

$$\varphi(z)=\lambda f(z)+(1-\lambda)g(z)=z-\sum_{n=2}^{\infty}c_nz^n,$$

where $c_n=\lambda a_n+(1-\lambda)b_n,\ n=2,3,\ldots$.

Apply Theorem 2.2, we can easily write

$$\sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)\left(n-\alpha-(n-1)\alpha\beta\right)c_n=\lambda\sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)\left(n-\alpha-(n-1)\alpha\beta\right)a_n$$

$$+(1-\lambda)\sum_{n=2}^{\infty}\left(1+(n-1)\gamma\right)\left(n-\alpha-(n-1)\alpha\beta\right)b_n\le\lambda(1-\alpha)+(1-\lambda)(1-\alpha)=1-\alpha.$$

This immediately completes the proof of Theorem 3.1.

Next we define the modified Hadamard product of the functions in the class T by

$$(f*g)(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n,$$

where $f(z)$ and $g(z)$ are functions defined by (1.2) and (3.1), respectively.

Theorem 3.2. *If each of the functions* $f(z)$ and $g(z)$ *is in the class* $TS^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1)$, $\gamma\in[0,1]$, *then* $\varphi\in TS^*C(\mu,\beta;\gamma)$, *where*

$$\varphi(z)=(f*g)(z)\ \text{ and }\ \mu\le 1-\frac{(1-\beta)(1-\alpha)^2}{(1+\gamma)\left(2-(1+\beta)\alpha\right)^2-(1+\beta)(1-\alpha)^2}.$$

Proof. From the Theorem 2.2, we have

$$\sum_{n=2}^{\infty}\frac{(1+(n-1))\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}a_n\le 1 \tag{3.2}$$

and

$$\sum_{n=2}^{\infty}\frac{(1+(n-1))\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}b_n\leq 1. \tag{3.3}$$

To complete the proof of Theorem 3.2, we have to find the largest μ such that

$$\sum_{n=2}^{\infty}\frac{(1+(n-1))\gamma)\left(n-\mu-(n-1)\mu\beta\right)}{1-\mu}a_n b_n\leq 1. \tag{3.4}$$

Applying the Cauchy-Schwarz inequality and using inequalities (3.2) and (3.3), we find

$$\sum_{n=2}^{\infty}\frac{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}\sqrt{a_n b_n}=$$

$$\sum_{n=2}^{\infty}\left[\frac{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}\right]^{1/2}\sqrt{a_n}\cdot\left[\frac{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}\right]^{1/2}\sqrt{b_n}\leq$$

$$\sum_{n=2}^{\infty}\frac{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}a_n\cdot\sum_{n=2}^{\infty}\frac{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}b_n\leq 1;$$

that is,

$$\sum_{n=2}^{\infty}\frac{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}{1-\alpha}\sqrt{a_n b_n}\leq 1. \tag{3.5}$$

It follows from (3.4) and (3.5) that, inequality (3.4) holds true if

$$\sqrt{a_n b_n}\leq\frac{(1-\mu)\left(n-\alpha-(n-1)\alpha\beta\right)}{(1-\alpha)\left(n-\mu-(n-1)\mu\beta\right)},\ n\geq 2,\ n\in\mathbb{N}. \tag{3.5}$$

On the other hand, since

$$\sqrt{a_n b_n}\leq\frac{1-\alpha}{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)},n\geq 2,\ n\in\mathbb{N},$$

(3.5) is satisfied if

$$\frac{1-\alpha}{(1+(n-1)\gamma)\left(n-\alpha-(n-1)\alpha\beta\right)}\leq\frac{(1-\mu)\left(n-\alpha-(n-1)\alpha\beta\right)}{(1-\alpha)\left(n-\mu-(n-1)\mu\beta\right)},\ n\geq 2,\ n\in\mathbb{N}.$$

Solve of the last inequality according to μ gives the following inequality

$$\mu\leq 1-\frac{(n-1)(1-\beta)(1-\alpha)^2}{(1+(n-1)\gamma)\left(n-(1+(n-1)\beta)\alpha\right)^2-(1+(n-1)\beta)(1-\alpha)^2},\ n\geq 2,\ n\in\mathbb{N}.$$

If we use that the function $h:\mathbb{N}\to\mathbb{R}$, defined by

$$h(n)=1-\frac{(n-1)(1-\beta)(1-\alpha)^2}{(1+(n-1)\gamma)\left(n-(1+(n-1)\beta)\alpha\right)^2-(1+(n-1)\beta)(1-\alpha)^2},\ n\geq 2,\ n\in\mathbb{N}$$

for all $\alpha,\beta\in[0,1),\ \gamma\in[0,1]$ is increasing, then we have

$$\mu\leq\min\left\{h(n):\ n\geq 2 \text{ and } n\in\mathbb{N}\right\}=h(2).$$

Thus, the proof of Theorem 3.2 is completed.

By taking $\gamma=0$ and $\gamma=1$ in Theorem 3.2, we can readily deduce the following results, respectively.

Corollary 3.1. *If each of the functions* $f(z)$ and $g(z)$ *is in the class* $TS^*(\alpha,\beta)$, $\alpha,\beta\in[0,1)$, *then* $\varphi\in TS^*C(\mu,\beta)$, *where*

$$\varphi(z)=(f*g)(z) \text{ and } \mu\le 1-\frac{(1-\beta)(1-\alpha)^2}{(2-(1+\beta)\alpha)^2-(1+\beta)(1-\alpha)^2}.$$

Corollary 3.2. *If each of the functions* $f(z)$ and $g(z)$ *is in the class* $TC(\alpha,\beta)$, $\alpha,\beta\in[0,1)$, *then* $\varphi\in TS^*C(\mu,\beta)$, *where*

$$\varphi(z)=(f*g)(z) \text{ and } \mu\le 1-\frac{(1-\beta)(1-\alpha)^2}{2(2-(1+\beta)\alpha)^2-(1+\beta)(1-\alpha)^2}.$$

By taking $\beta=0$ in Corollary 3.1 and 3.2, respectively, we obtain the following results.

Corollary 3.3. *If each of the functions* $f(z)$ and $g(z)$ *is in the class* $TS^*(\alpha)$, $\alpha\in[0,1)$, *then* $\varphi\in TS^*C(\mu)$, *where*

$$\varphi(z)=(f*g)(z) \text{ and } \mu\le\frac{2-\alpha^2}{3-2\alpha}.$$

Corollary 3.4. *If each of the functions* $f(z)$ and $g(z)$ *is in the class* $TC(\alpha)$, $\alpha\in[0,1)$, *then* $\varphi\in TS^*C(\mu)$, *where*

$$\varphi(z)=(f*g)(z) \text{ and } \mu\le\frac{6-4\alpha}{\alpha^2-6\alpha+7}.$$

Remark 3.1. Further consequences of the properties given by Theorem 3.1 and Theorem 3.2 can be obtained for each of the classes studied by earlier studies, by specializing the various parameters involved.

4. Distortion theorems for the fractional operators

In this section, we will give distortion theorems for the functions belonging to the class $TS^*C(\alpha,\beta;\gamma)$. Each of these theorems would involve certain integral of fractional calculus, which are defined as follows (see for details, [14- 16]).

Definition 4.1. *The fractional integral of order* δ *is defined by*

$$D_z^{-\delta}f(z)=\int_0^z\frac{f(t)}{(z-t)^{1-\delta}}dt,\ z\in\mathbb{C},\ \delta>0,$$

where $f(z)$ *is an analytic function in a simply-connected region of the complex plane containing the origin, and the multiplicity of* $(z-t)^{\delta-1}$ *is removed by requiring* $\ln(z-t)$ *to be real* $z-t>0$.

Definition 4.2. *The fractional derivative of order* δ *is defined by*

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\delta}}dt,\ z\in\mathbb{C},\ \delta\in[0,1),$$

where $f(z)$ *is constrained, and the multiplicity of* $(z-t)^{-\delta}$ *is removed, as in Definition 4.1.*

Definition 4.3. *Under the hypotheses of Definition 4.1, the fractional derivative of order* $n+\delta$ *is defined by*

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z),\ \delta \in [0,1),\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Theorem 4.1. *Let* $f \in TS^*C(\alpha,\beta;\gamma),\ \alpha,\beta \in [0,1),\ \gamma \in [0,1]$. *Then,*

$$\begin{aligned} &|z|^{1+\delta}\left(\frac{1}{\Gamma(2+\delta)} - \frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(3+\delta)}|z|\right) \\ &\le \left|D_z^{-\delta} f(z)\right| \le |z|^{1+\delta}\left(\frac{1}{\Gamma(2+\delta)} + \frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(3+\delta)}|z|\right) \end{aligned} \tag{4.1}$$

for $\delta > 0$ *and for all* $z \in U$.
The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{(1+\gamma)(2-(1+\beta)\alpha)} z^2,\ z \in U. \tag{4.2}$$

Proof. Suppose that $f \in TS^*C(\alpha,\beta;\gamma),\ \alpha,\beta \in [0,1),\ \gamma \in [0,1]$. Then, from the Definition 4.1, we have

$$D_z^{-\delta} f(z) = z^{1+\delta}\left(\frac{\Gamma(2)}{\Gamma(2+\delta)} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} a_n z^n\right). \tag{4.3}$$

Since the function $\psi : \mathbb{N} \to \mathbb{R}$, defined by

$$\psi(n) = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)},\ n \ge 2,\ n \in \mathbb{N},\ \delta > 0,$$

is decreasing, by using the triangle inequality, we find from (4.3) and (2.7) that

$$\left|D_z^{-\delta} f(z)\right| \le |z|^{1+\delta}\left(\frac{\Gamma(2)}{\Gamma(2+\delta)} + \frac{(1-\alpha)\Gamma(3)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(3+\delta)}|z|\right).$$

This completes the proof of the right hand side of the inequality (4.1).
From (4.3), we write

$$\left|D_z^{-\delta} f(z)\right| \ge |z|^{1+\delta}\left(\frac{\Gamma(2)}{\Gamma(2+\delta)} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} a_n |z|^n\right).$$

Similarly, from here, we obtain

$$\left|D_z^{-\delta} f(z)\right| \ge |z|^{1+\delta}\left(\frac{\Gamma(2)}{\Gamma(2+\delta)} - \frac{(1-\alpha)\Gamma(3)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(3+\delta)}|z|\right).$$

But, this is the same of the left hand side of the inequality (4.1).

Further, easily see that the equality in (4.1) is satisfied by the function $f(z)$ given by (4.2).

Thus, the proof of Theorem 4.1 is completed.

The proofs of the following theorems are very similar to the proof of Theorem 4.1, so the details of the proofs may be omitted.

Theorem 4.2. *Let* $f \in TS^*C(\alpha,\beta;\gamma),\ \alpha,\beta \in [0,1),\ \gamma \in [0,1]$. *Then,*

$$|z|^{1-\delta}\left(\frac{1}{\Gamma(2-\delta)}-\frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(3-\delta)}|z|\right)$$
$$\leq\left|D_z^{\delta} f(z)\right|\leq|z|^{1-\delta}\left(\frac{1}{\Gamma(2-\delta)}+\frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(3-\delta)}|z|\right)$$

for $\delta\in[0,1)$ *and for all* $z\in U$. *The result is sharp for the function* $f(z)$ *given by (4.2).*

Theorem 4.3. *Let* $f\in TS^{*}C(\alpha,\beta;\gamma),\ \alpha,\beta\in[0,1),\ \gamma\in[0,1]$. *Then,*

$$|z|^{-\delta}\left(\frac{1}{\Gamma(1-\delta)}-\frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(2-\delta)}|z|\right)$$
$$\leq\left|D_z^{1+\delta} f(z)\right|\leq|z|^{-\delta}\left(\frac{1}{\Gamma(1-\delta)}+\frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)\Gamma(2-\delta)}|z|\right)$$

for $\delta\in[0,1)$ *and for all* $z\in U$. *The result is sharp for the function* $f(z)$ *given by (4.2).*

By taking $\delta=0$ in Theorem 4.2, we can readily deduce the following corollary.

Corollary 4.1. *If* $f\in TS^{*}C(\alpha,\beta;\gamma),\ \alpha,\beta\in[0,1),\ \gamma\in[0,1]$, *then*

$$|z|\left(1-\frac{1-\alpha}{(1+\gamma)(2-(1+\beta)\alpha)}|z|\right)\leq|f(z)|\leq|z|\left(1+\frac{1-\alpha}{(1+\gamma)(2-(1+\beta)\alpha)}|z|\right)$$

for all $z\in U$. *The result is sharp for the function* $f(z)$ *given by (4.2)*

If, we set $\delta=0$ in Theorem 4.3, we arrive at the following corollary.

Corollary 4.2. *If* $f\in TS^{*}C(\alpha,\beta;\gamma),\ \alpha,\beta\in[0,1),\ \gamma\in[0,1]$, *then*

$$1-\frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)}|z|\leq|f'(z)|\leq1+\frac{2(1-\alpha)}{(1+\gamma)(2-(1+\beta)\alpha)}|z|$$

for all $z\in U$. *The result is sharp for the function* $f(z)$ *given by (4.2)*

Remark 4.1. Numerous consequences of the distortion properties given by Corollary 4.1 and Corollary 4.2 can be obtained for each of the classes studied by earlier studies.

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