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# Some Inequalities for the Frobenius Norm of Complex Matrix

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**Abstract** In this paper, we use the existing conclusions of unitary invariant norm inequalities to present some inequalities under certain conditions of the Frobenius norm of the complex matrix, at the same time, we extend the corresponding results.

**Keywords** unitary invariant norm, Frobenius norm, inequalities

## 1. Introduction

For a complex number $x = a + ib$, where $a, b$ are all real, we write $\operatorname{Re} x = a$, $\operatorname{Im} x = b$, as usual, let $C^{m\times n}$ be the set of $m \times n$ complex matrices, $U_n$ be the set of all $n \times n$ unitary matrices, $A$ be an $n \times n$ matrix, denote the eigenvalues of $A$ by $\lambda_i(A)$, the singular values of $A$ by $\sigma_i(A)$, the trace of $A$ by $trA$, the associate matrix of $A$ by $A^H$, and the Frobenius norm of $m \times n$ complex matrix by

$$\|A\|_F = \left(trA^H A\right)^{1/2} = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}, \ A = (a_{ij}) \in M^{m\times n}.$$

A great deal of work on the topic has been done by a number of authors [2–6], in 1979, *Marshall* and *Olkin* gave the following inequality [3]

$$\|A - B(UV^*)\|_E \le \|A - B\Gamma\|_E \tag{1.1}$$

Where $A, B$ are complex matrices, $U, V, \Gamma$ are unitary matrices and $B^* A = U(B^* A)_\sigma V^*$.

In1999, Wang boying extended the (1.1), made it also valid in unitary invariant norm [4], and got the equality

$$\|A_\sigma - B_\sigma\|_{ui} \le \|A - UBV\|_{ui} \le \|A_\sigma + B_\sigma\|_{ui} \tag{1.2}$$

Where $A, B$ are complex matrices, $U, V$ are unitary matrices, $\|\cdot\|_{ui}$ is the unitary invariant norm.

The purpose of this paper is to use the existing conclusions to present some inequalities under certain conditions of the Frobenius norm of the complex matrix, and extend the results of Wang boying.

## 2. Main Results

**Theorem 1.** Let $A$ and $B$ be $n \times n$ complex matrices, and $C$ be $m \times m$ complex matrix, $W$ be $m \times n$ complex matrix,

(1) If $\mathrm{Re}\,tr\left(A\left(B^H - W^H C^H W\right)\right) \geq 0$ and $\left\|W^H CW\right\|_F \geq \|B\|_F$, then $\|A-B\|_F \leq \left\|A - W^H CW\right\|_F$ ;

(2) If $\mathrm{Re}\,tr\left(A\left(B^H - W^H C^H W\right)\right) \leq 0$ and $\left\|W^H CW\right\|_F \leq \|B\|_F$, then $\|A-B\|_F \geq \left\|A - W^H CW\right\|_F$ ;

(3) If $\mathrm{Re}\,tr\left(A\left(B^H - W^H C^H W\right)\right) = 0$ and $\left\|W^H CW\right\|_F = \|B\|_F$, then $\|A-B\|_F = \left\|A - W^H CW\right\|_F$ .

**Proof**. We first show (1).

$$Re\,tr\left(A(B^H - W^H C^H W)\right) \geq 0 \Leftrightarrow 2Re\,tr(AB^H) \geq Re\,tr(AW^H C^H W)$$

$$\Leftrightarrow tr(AB^H + B^H A) \geq tr(AW^H C^H W + A^H W^H CW)$$

$$\left\|W^H CW\right\|_F \geq \|B\|_F \Leftrightarrow tr\left[(W^H CW)^H (W^H CW)\right] \geq tr(B^H B)$$

$$\Leftrightarrow tr(W^H C^H W W^H CW) \geq tr(B^H B)$$

Thus, we have

$$\begin{aligned}\|A - W^H CW\|_F^2 &= tr(A - W^H CW)^H (A - W^H CW)\\ &= tr(A^H A + W^H C^H W W^H CW - A^H W^H CW - AW^H C^H W)\\ &\geq tr(A^H A + B^H B - AB^H - B^H A)\\ &= tr(A-B)^H (A-B) = \|A-B\|_F^2 .\end{aligned}$$

Then we show (2).

$$Re\,tr\left(A(B^H - W^H C^H W)\right) \leq 0 \Leftrightarrow 2Re\,tr(AB^H) \leq Re\,tr(AW^H C^H W)$$

$$\Leftrightarrow tr(AB^H + B^H A) \leq tr(AW^H C^H W + A^H W^H CW)$$

$$\left\|W^H CW\right\|_F \leq \|B\|_F \Leftrightarrow tr\left[(W^H CW)^H (W^H CW)\right] \leq tr(B^H B)$$

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The (1) and (2) yield the right of (3). The proof is completed.

If $W$ is any unitary matrix , and its *Frobenius* norm is unitary invariant norm, then,

$\left\|W^H CW\right\|_F = \|C\|_F \leq \|B\|_F$ , it satisfy the conditions of Theorem1, thus, the following result holds.

**Corollary 1.** Let $A$, $B$ and $C$ be $n \times n$ complex matrices, $W$ be $n \times n$ unitary matrix, then

(1) If $Re\,tr\left(A(B - W^H CW)\right) \geq 0$ and $\|C\|_F \geq \|B\|_F$, then $\|A-B\|_F \leq \|A - W^H CW\|_F$ ;

(2) If $Re\,tr\left(A(B - W^H CW)\right) \leq 0$ and $\|C\|_F \leq \|B\|_F$, then $\|A-B\|_F \geq \|A - W^H CW\|_F$ ;

(3) If $Re\, tr\left(A(B-W^H CW)\right)=0$ and $\| C\|_F = \| B\|_F$, then $\| A-B\|_F = \| A-W^H CW\|_F$.

**Remark 1.** If $C=B$ in Corollary1 , it is easy to see the following inequalities hold.

(1) If $Re\, tr\left(A(B-W^H BW)\right)\geq 0$, then

$$\| A-B\|_F \leq \| A-W^H CW\|_F \; ; \tag{2.1}$$

(2) If $Re\, tr\left(A(B-W^H BW)\right) \leq 0$, then

$$\| A-B\|_F \geq \| A-W^H CW\|_F \; ; \tag{2.2}$$

(3) If $Re\, tr\left(A(B-W^H BW)\right)=0$, then

$$\| A-B\|_F = \| A-W^H CW\|_F . \tag{2.3}$$

The (2.1) is the inequality of the theorem 2.4 in [5].

**Theorem 2.** Let $A$, $B$ and $C$ be $n\times n$ complex matrices, $W$ be $n\times n$ unitary matrix, if $Re\, tr\left(AW^H C^H W + A^H B\right)\geq 0$ and $\| C\|_F \leq \| B\|_F$, then

$$\| A-W^H CW\|_F \leq \| A+B\|_F . \tag{2.4}$$

**Proof.** $\| C\|_F \leq \| B\|_F \Leftrightarrow tr(C^H C)\leq tr(B^H B)$,

$$\begin{aligned} Re\, tr(AW^H C^H W - A^H B)\geq 0 &\Leftrightarrow tr(AW^H C^H W + A^H W^H CW + A^H B + B^H A)\geq 0 \\ &\Leftrightarrow tr(A^H B + B^H A)\geq -tr(AW^H C^H W + A^H W^H CW) \end{aligned}$$

thus

$$\begin{aligned} \| A-W^H CW\|_F^2 &= tr(A-W^H CW)^H (A-W^H CW) \\ &= tr(A^H A + W^H C^H CW - A^H W^H CW - AW^H C^H W) \\ &= tr(A^H A + C^H C) - tr(A^H W^H CW + AW^H C^H W) \\ &\leq tr(A^H A + B^H B + A^H B + B^H A) \\ &= tr(A+B)^H (A+B) = \| A+B\|_F^2 . \end{aligned}$$

The proof is completed.

**Corollary 2** Let $A$ and $B$ be positive semidefinite matrices, $W$ be any unitary matrix, then

$$\| A-W^H BW\|_F \leq \| A+B\|_F . \tag{2.5}$$

**Proof.** For $A\geq 0$, $B\geq 0$, notice that $W^H BW\geq 0$, then

$$\begin{aligned} tr(AW^H BW) &= \sum_{i=1}^{n}\lambda_i(AW^H BW) \geq \sum_{i=1}^{n}\lambda_i(W^H BW)\lambda_{n-i+1}(A) \\ &= \sum_{i=1}^{n}\lambda_i(B)\lambda_{n-i+1}(A)\geq 0 , \end{aligned}$$

$$tr(A^H B) = tr(AB) = \sum_{i=1}^{n}\lambda_i(AB) \geq \sum_{i=1}^{n}\lambda_i(A)\lambda_{n-i+1}(B) \geq 0 ,$$

Hence, we get

$$Re\, tr(AW^H C^H W + A^H B) = tr(AW^H BW) + tr(A^H B) \geq 0.$$

Particularly, for $C = B$, by Theorem2, the inequality(2.5) is completed.

**Remark 2.** The Theorem2 in [4] point that : If $A, B$ are complex matrices,
$A_\sigma = diag(\sigma_1(A), \sigma_2(A), \cdots, \sigma_n(A))$, $B_\sigma = diag(\sigma_1(B), \sigma_2(B), \cdots, \sigma_n(B))$ ,where $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$, $\sigma_1(B) \geq \cdots \geq \sigma_n(B) \geq 0$. Let $U$ , $V$ are unitary matrices, then

$$\| A - VBU \| \leq \| A_\sigma + B_\sigma \|,$$

Where $\| \cdot \|$ is the unitary invariant norm.

Particularly, let unitary invariant norm be Frobenius norm such that $V = U^H$ , then, we get

$$\| A - U^H BU \|_F \leq \| A_\sigma + B_\sigma \|_F \tag{2.6}$$

if $A$ and $B$ are diagonal and positive semidefinite matrices, then ,there is an unitary matrix $P$ , leads to $P^H AP = \Lambda_1$ , $P^H BP = \Lambda_2$ , where $\Lambda_1 = diag(\sigma_{j_1}(A), \sigma_{j_2}(A), \cdots, \sigma_{j_n}(A))$ , $\Lambda_2 = diag(\sigma_{k_1}(B), \sigma_{k_2}(B), \cdots, \sigma_{k_n}(B))$.

So by Theorem 2, we get

$$\begin{aligned}
\| A + B \|_F^2 &= \| P^H (A+B) P \|_F^2 \\
&= \| diag((\lambda_{j_1}(A) + \lambda_{k_1}(B)) + \cdots + (\lambda_{j_i}(A) + \lambda_{k_i}(B))) \|_F^2 \\
&= \| \Lambda_1 + \Lambda_2 \|_F^2 = \sum_{i=1}^{n} (\lambda_{j_i}(A) + \lambda_{k_i}(B))^2 \\
&= \sum_{i=1}^{n} \lambda_{j_i}{}^2(A) + \sum_{i=1}^{n} \lambda_{k_i}{}^2(B) + 2\sum_{i=1}^{n} \lambda_{j_i}(A)\lambda_{k_i}(B) \\
&\leq \sum_{i=1}^{n} \lambda_i{}^2(A) + \sum_{i=1}^{n} \lambda_i{}^2(B) + 2\sum_{i=1}^{n} \lambda_i(A)\lambda_i(B) \\
&= \sum_{i=1}^{n} (\lambda_i(A) + \lambda_i(B))^2 = \| A_\sigma + B_\sigma \|_F^2,
\end{aligned}$$

Thus,

$$\| A - U^H BU \|_F \leq \| A + B \|_F \leq \| A_\sigma + B_\sigma \|_F.$$

Obviously, the Corollary2 extended the conclusions of [4].

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