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On Maximal Injective Subalgebras of Tensor Products of Real W*-Algebras

Nurillaev Muzaffar Eshnazarovich

Tashkent State Pedagogical University after Nizami, The faculty of physics-mathematics, the department of General mathematics

AbstractLet R_1 be an injective real W*-algebra and R_2 be a real W*-algebra with separable predual. It is proven that if Q_2 is a maximal injective real W*-subalgebra of R_2, then $R_1\overline{\otimes}Q_2$ is a maximal injective real W*-subalgebra of $R_1\overline{\otimes}R_2$. This partly answers a question of Popa for real W*-algebras.

Keywordsreal W*-algebras, injective W*-algebras, tensor product of W*-algebras.

1. Introduction

Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on *H*. A complex *-subalgebra $M \subset B(H)$ with the identity **1** is called a *W*-algebra*, if it is weakly closed. A real *-subalgebra $R \subset B(H)$ with **1** is called areal *W*-algebra*, if it is weakly closed and $R \cap iR = \{0\}$. A complex or real W*-algebra A is called *injective*, if there is a projection $P : B(H) \to A$ such that $\|P\| = 1$ and P(**1**) = **1** (see [1], [2] for complex case and [3] for real case). In the present time complex and real injective factors have been investigated wellenough. Compare with injective factors, non-injective factors are far from being understood. A standard method of investigation in the study of general factors is to study the injective subalgberas of these factors. Along this line, we have R. Kadison's question (Problem 7, [4]): does each self-adjoint operator in a II_1 factor lie in some hyperfinitesubfactor? Recall that a complex or real W*-algebra *A* is called *hyperfinite*, if there is an increasing sequence $\{A_n\}$ of finite dimensional W*-subalgebras with **1**, such that the union of them is weakly dense in *A*. Since each separable abelian W*-algebra is generated by a single self-adjoint operator, Kadison's question has an equivalent form: is each separable abelian W*-algebra of a II_1 factor contained in some hyperfinitesubfactor? In 1983, SorinPopa gave a negative answer. He built an example, in which an abelian W*-algebra can be embedded in a factor of type II_1 as a maximal injective W*-subalgebra, i.e. it is not an embedding as a hyperfinitesubfactor. Recall that a complex or real W*-subalgebra is called *maximal injective* if it is injective and if it is maximal with respect to inclusion in the set of all injective W*-subalgebras.

In [5] S.Popa formulated the following two problems, connecting with this problem: 1) does each II_1-factor contain a hyperfinite subfactor as a maximal injective W*-subalgebra? 2) if $N_1 \subset M_1$ and $N_2 \subset M_2$ are maximal injective W*-subalgebras, is $N_1 \otimes N_2$ maximal injective W*-subalgebra in $M_1 \otimes M_2$? In the present paper we answer the second question of Popa for real W*-algebras and it was obtained as a result in the following particular case: if R_1 is an injective real W*-algebra, R_2 is a real W*-algebra with separable predual and Q_2 is a maximal injective real W*-subalgebra of R_2, then $R_1\overline{\otimes}Q_2$ is a maximal injective real W*-subalgebra of $R_1\overline{\otimes}R_2$. Moreover, we prove the real analogue Ge-Kadison's splitting theorem for finite case.

2. Real Analogue of Ge-Kadison's Splitting Theorem

Firstly, we prove some auxiliary results.

Lemma 2.1. Let *R* be a real W*-algebra and *F*, *Q* be real W*-subalgebras of *R*, such that $F\subset Q$. If *E* is a conditional expectation from *R* onto *Q* (i.e. it is positive, linear mapping with $E(abc)=aE(b)c$, for all $a,c\in Q$, $b\in R$), then *E* induces a conditional expectation from $F'\cap R$ onto $F'\cap Q$.

Proof. Let $x\in F'\cap R$ and $y\in F$. Then $xy=yx$ and apply the conditional expectation *E* to both sides of $xy=yx$. Since $F\subset Q$, then $E(x)y=yE(x)$. Thus $E(x)\in F'\cap Q$. According to $F'\cap Q\subset F'\cap R$, *E* is a conditional expectation from $F'\cap R$ onto $F'\cap Q$ when *E* is restricted on $F'\cap R$. $\square$

Lemma 2.2. Let F_i, $i=1,\ 2$ be a real W*-algebra, acting on a complex Hilbert space H_i and S_i be a real W*-subalgebra of F_i, $i=1,\ 2$. Let F be a real W*-algebra such that $S_1\overline{\otimes}S_2\subset F\subset F_1\overline{\otimes}F_2$. If there is a conditional expectation $E:F_1\overline{\otimes}F_2\to F$, then *E* induces a conditional expectation from $(S_1'\cap F_1)\overline{\otimes}F_2$ onto $((S_1'\cap F_1)\overline{\otimes}F_2)\cap F$.

Proof.By Lemma 1.7 in [6] we have

$$(U(F_1)\overline{\otimes}U(F_2))\cap(U(S_1)\overline{\otimes}\square 1)'=(U(F_1)\overline{\otimes}U(F_2))\cap(U(S_1)'\overline{\otimes}B(K))=$$
$$=(U(S_1)'\cap U(F_1))\overline{\otimes}U(F_2),$$

where $U(\circ)$ is the enveloping W*-algebra of the real W*-algebra "$\circ$". Since

$$U(\circ_1\overline{\otimes}\circ_2)=U(\circ_1)\overline{\otimes}U(\circ_2) \qquad (2.1)$$

then we obtain $(F_1\overline{\otimes}F_2)\cap(S_1\overline{\otimes}\square 1)'=(S_1'\cap F_1)\overline{\otimes}F_2$. By Lemma 2.1, *E* induces a conditional expectation from $(S_1'\cap F_1)\overline{\otimes}F_2$ onto $((S_1'\cap F_1)\overline{\otimes}F_2)\cap F$. $\square$

From Lemma 2.2 we obtain the following corollary.

Corollary 2.3. Assume the conditions of Lemma 2.2 and $S_1'\cap F_1=\square 1$. Let $L_2=\{x\in F_2:\ 1\otimes x\in F\}$. Then *E* induces a conditional expectation from F_2 onto L_2.

Now using Lemma 2.2 and Corollary 2.3 we prove the real analogue of Ge-Kadison's splitting theorem for finite real W*-algebras.

Theorem 2.4. If R_1 is a finite real factor and R_2 is a finite realW*-algebra, and R is a real W*-subalgebraof $R_1\overline{\otimes}R_2$ which contains $R_1\otimes\mathbf{R}\cdot\mathbf{1}$, then $R=R_1\overline{\otimes}Q_2$ for some Q_2, a real W*-subalgebra of R_2.

Proof.Let *Q* be a realW*-algebra such that $R_1\overline{\otimes}\square 1\subset Q\subset R_1\overline{\otimes}R_2$. Since R_1 and R_2 are finite, then there is a normal conditional expectation *E* from $R_1\overline{\otimes}R_2$ onto Q. By Corollary 2.3 *E* induces a conditional expectation, denoted by E_2 from R_2 onto $Q_2:=\{T\in R_2:1\otimes T\in Q\}$. Now for any $x\in R_1$, $y\in R_2$, we have

$$E(x\otimes y)=x\otimes E_2(y)\in R_1\overline{\otimes}Q_2.$$

Since E is normal, $Q = E(R_1 \overline{\otimes} R_2) \subset R_1 \overline{\otimes} Q_2$. Since $Q \supset R_1 \overline{\otimes} Q_2$, $Q = R_1 \overline{\otimes} Q_2$. □

3. Maximal Injective Subalgebras of Tensor Product of Real W*-Algebras.

Now we shall prove one more auxiliary result.

Lemma 3.1. Let A be an abelian real W*-algebra and F_2 be a minimal injective real W*-algebra extension of real W*-algebra Q. Suppose F_2 has separable predual. If F is an injective real W*-algebra such that

$$A \overline{\otimes} Q \subset F \subset A \overline{\otimes} F_2 \qquad (3.1)$$

then $F = A \overline{\otimes} F_2$.

Proof.By (2.1) and (3.1) we have $U(A) \overline{\otimes} U(Q) \subset U(F) \subset U(A) \overline{\otimes} U(F_2)$. By Lemma 2.4 in [6], we have got $U(F) = U(A) \overline{\otimes} U(F_2)$.

Since $U(A) \overline{\otimes} U(F_2) = U(A \overline{\otimes} F_2)$ and $A \subset F \subset F_2$, then from $U(F) = U(A \overline{\otimes} F_2)$ we can obtain $F = A \overline{\otimes} F_2$. □

The main result of the work is the following theorem.

Theorem 3.2. Let R_1 be an injective real W*-algebra and R_2 be a real W*-algebra with separable predual. If Q_2 is a maximal injective real W*-subalgebra of R_2, then $R_1 \overline{\otimes} Q_2$ is a maximal injective real W*-subalgebra of $R_1 \overline{\otimes} R_2$.

Proof.Wecan assume that R_1 and R_2 are real W*-algebras, acting on complex Hilbert spaces H and K, respectively. Then K is a separable Hilbert space. Let A be the center of R_1. Suppose Q is an injective real W*-algebra such that $Q_2' \supset Q \supset R_1' \overline{\otimes} R_2'$. Since Q' is an injective real W*-subalgebra of $R_1' \overline{\otimes} Q_2'$, then there is a conditional expectation E from $R_1' \overline{\otimes} Q_2'$ onto Q'. By Lemma 2.2, E induces a conditional expectation from $A \overline{\otimes} Q_2'$ onto $F := (A \overline{\otimes} Q_2') \cap Q'$. So, F is an injective real W*-algebra such that $A \overline{\otimes} Q_2' \supset F \supset A \overline{\otimes} R_2'$. Since Q_2 is a maximal injective real W*-subalgebra of R_2, then Q_2' is a minimal injective real W*-subalgebraic extension of R_2'. By Lemma 3.1 we have $F = A \overline{\otimes} Q_2'$. Thus □ $1 \overline{\otimes} Q_2' \subset F \subset Q'$. So, $Q' = R_1' \overline{\otimes} Q_2'$ and $Q = R_1 \overline{\otimes} Q_2$. □

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