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***Journal of Scientific and Engineering Research*, 2017, 4(8):374-378**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Pregroups from Length Functions

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Abstract Stallings [1] in 1971 introduced the concept of a pregroup. Subsequent work has been done by Hoare [2], Nesayef [3], Chiswell [4], and many others. Five axioms were originally introduced by Stallings [1], namely P1, P2, P3, P4, and P5. It has been proved in [5] that P3 is a consequence of the other axioms.Further development was conducted by Stalling in [6].

Keywords Archimedean Elements, Defined Product of Elements, Length Functions, Pregroup, Universal Group

1. Introduction

In section one, we introduce the concept of length function and list all the other axioms of Length Function which are needed in the latter sections. We introduce the definitions and some important properties of pregroups. In section two, we introduce a new axiom called P6 and proved that some axioms are equivalent to the other ones.

We also investigate some basic properties of Pregroups restricted to P6. Finally we showed that the universal group of a Pregroup satisfying P6 has a length function given by Lyndon [7].

Length Functions and Pregroups

Definition 2.1 : A Length Function | | on a group G, is a function given each element x of G a real number $|x|$, such that for all $x , y , z \in G$, the following axioms are satisfied.

$A1' \, |e| = 0$, e is the identity elements of G .

$A2 \; |x^{-1}| = |x|$

$A4 \; d(x,y) < \; d(y,z) \Longrightarrow d(x,y) = d(x,z)$, where $d(x,y) = \frac{1}{2} \, (\, |x| + |y| - |xy^{-1}|$

Lyndon showed that A4 is equivalent to $d(x,y) \geq \min\{ \, d(y,z), d(x,z)\}$ and to

$d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m$.

$A1', A2$ and $A4$ imply $|x| \geq d(x,y) = d(y,x) \geq 0$

Assuming, A2 and A4 only, it is easy to show that:

i. $e \; d(x,y) \geq |e|$
ii. $|x| \; \geq |e|$
iii. $d(x,y) \leq |x| - \frac{1}{2}|e|$.

$A3$ state that $d(x,y) \geq 0$ is deductible from $A1', A2$ and $A1'$ is a weaken version of the axiom:

$A1 |x| = 0$if and only if $x = 1$ in G .

$N1^{*} \; G \; is \; general \; by \; \{x \in G : \; |x| \leq 1\}$

Definition 2.2: A pregroup is a set P containing an element called the identity element of P, denoted by 1, a subset D of, PXP and a mapping D → P, when (x , y) x y together with a map i : P → P where i (x) = x^{-1}, satisfying the following axioms: (we say that x y is defined if (x , y) ∈ D, i.e. x y∈ P)

P1. For all x ∈ P, 1x and x1 are defined and 1x = x1 = x.

P2. For all x ∈P , x^{-1} x = x x^{-1} = 1.

P3. For all x ,y∈P, if x y is defined , then y^{-1} x is defined and $(x y)^{-1}$ = y x.

P4. Suppose that x, y, z ∈ P. If x y and y z are defined, then x (y z) is defined, is which case

x (y z) = (x y) z.

P5. If w, x, y, z ∈P, and if w x, x y, y z, are all defined them either w (x y) or (x y) z is defined.

2.1 The axiom P6

In this section we shall restrict our attention to a special type of pregroup, which satisfies a certain axiom. To do this we introduce the following theorem, which was introduced in [3].

Theorem 2.1: The following two statements are equivalent in P.

P6(1) if (x_1 , x_2) is reduced and $x_1 a , a^{-1} x_2$ are both defined , then $a \in A_0$

P6(2) if (x_1 , x_2) is reduced and $(a x_1)\, x_2$ is defined for $a \in P$ then $a x_1 \in A_0$

Proof suppose (x_1 , x_2) is reduced and let $(a x_1)\, x_2$ is defined for some $a \in P$.

$x_1 (a x_1)^{-1}$, $(a x_1)\, x_2$are both defined so $a x_1 \in A_0$, so P6 (1) → P6(2).

Conversely, suppose (x_1 , x_2) is reduced and $x_1 a , a^{-1} x_2$ are both defined for some $a \in P$

Since (x_1 , x_2) is reduced, then $(x_1 a , a^{-1} x_2)$ is reduced

Since $x_1^{-1}(x_1 a)$ is defined and = a, and $\{x_1^{-1}(x_1 a)\} a^{-1} x_2$ is defined and = x_2 , then by P6 (2) $x_1^{-1}(x_1 a) \in A_0$, i . e. $a \in A_0$.

Therefore P6 (1) ⇔ P6 (2)

We denote the equivalent statements P6(1) and P6(2) in theorem 2.1 by P6, and the pregroup which satisfies P6, by P*. Then we shall define a length function on $U\,(P^*)$, the universal group of the pergroup P*.

Before defining length function on $U\,(P^*)$we shall introduce the following result. The following theorem generalizes the condition P6 (2).

Theorem 2.2: Let $a_{n-1} , \dots , a_1$ be any sequence, and $x_1 , \dots , x_n$ be reduced, both on P^*. If $a_{n-1} , \dots , a_1 x_1 , \dots , x_n$ is defined, then $a_{n-1} , \dots , a_1 x_1 , \dots , x_n \in A_0$, $n \geq 2$

Proof The only way which $a_{n-1} , \dots , a_1 x_1 , \dots , x_n$ is defied is by $[a_{n-1} , \dots , a_1 x_1 , \dots , x_n] x_n$ being defined.

Then also by theorem 2.2 either $[(a_{n-1} , \dots , a_1 x_1 , \dots , x_{n-2}) x_{n-1}] x_n$ is defied, so by P6(2), $(a_{n-1} , \dots , a_1 x_1 , \dots , x_{n-2}) x_{n-1} \in A_0$, or $[\, a_{n-1}(a_{n-2} \dots a_1 x_1 \ \dots\ x_{n-1})] x_n$ is defined, where $x_0 = 1$.

Since $(a_{n-2} , \dots , a_1 x_1 , \dots , x_{n-1}) x_n$ is not defined by theorem 2.5, then by P6(2) $a_{n-1}\, (a_{n-2} \ \dots\ a_1 x_1 \ \dots\ x_{n-1}) \in A_0$.

Theorem 2.3: Let $U\,(P^*)$ be the universal group of a pergroup P^* and left $g, h \in (P^*)$ by given $g = x_1 \dots x_n$, $h = y_1 \dots y_m$, $m, n \geq 2$in reduced forms. Let $a_i = x_{n-i+1} \dots . (x_n y_m^{-1}) \dots y_{m-i+1}^{-1}$ be defined for $1 \leq i \leq s$ for some $s < m , s \leq n$. If $a_s y_{m-s}^{-1}$ is defined then $a_i \in A_0$ for all $\leq s$, hence by symmetry if $s < n$ and $x_{n-s} a_s$ is defined, then $a_i \in A_0$.

Proof $a_i = x_{n-i+1} a_{i-1}\, y^{-1}_{m-i+1}$ for $1 \le i < s$, which $a_0 = 1$ then by theorem 2.5 either $x^{-1}_{n-i+1}(a_{i-1}\, y^{-1}_{m-i+1})$ is defined (1)

or$(\,x^{-1}_{n-i+1}\, a_{i-1})\, y^{-1}_{m-i+1}$ is defined (2)

If (1) holds , then $[\,x^{-1}_{n-i+1}\,(a_{i-1}\, y^{-1}_{m-i+1})]\, y^{-1}_{m-i}$ is defined

Then $a_{i-1}\, y^{-1}_{m-i+1}$, y^{-1}_{m-i} is reduced, so by P6 (2) $[\,x_{n-i+1}(a_{i-1}\, y^{-1}_{m-i+1})] = a_i \in A_0$

If (2) holds, then $[(\,x_{n-i+1} a_{i-1})\, y^{-1}_{m-i+1}]\, y^{-1}_{m-i}$ is defined.

Since y^{-1}_{m-i+1} , y^{-1}_{m-i} is reduced, then by P6 (2) ,$(\,x_{n-i+1} a_{i-1})\, y^{-1}_{m-i+1} = a_i \in A_0$

Corollary 2.1: In theorem 2.3, $x_{n-s}\, a_s\, (\,a_s y^{-1}_{m-s})$ is defined if and only if $(\,x_{n-s}\, a_s)\, a_s y^{-1}_{m-s}$ is defined.

Proof Since $a_s \in A_0$ in either case. Then $x_{n-s}\, a_s$ and $a_s y^{-1}_{m-s}$ are both defined.

By P4, the result follows.

From theorems 2.1 and 2.2, with the same notation, we have shown that:

Corollary 2.2: $gh^{-1} = x_1 \dots . x_{n-s} a_s y^{-1}_{m-s} \dots . y^{-1}_{m-s}$ is reduced, if and only if $a_s \notin A_0$.

Proof: See [3].

2.2 P* and Length Function

Theorem 2.4: $|\quad| : U(P^*) \to \mathbb{R}$ given in definition [3] is a length function on $U(P^*)$.

Proof See [3].

3. Pregroups from Length Functions

We now consider the converse of theorem 2.4, that is, we have a length function on a group G such that G is generated by elements of length zero and one, then these elements, of length 0,1 from a Pregroup P* satisfying P6, and $G \cong U(P^*)$ preserves length.

$P^* = \{x \in G : |x| \le 1\}$

$|1| = 0$, so$1 \in P^*$

For $x \in P^*$, $|x| = 0{,}1$ and by A2 ,$|x| = |x^{-1}|$

So $|x^{-1}| = 0{,}1$ and $x^{-1} \in P^*$

The product xy of two elements x and y of P^* is defined if and only if $|xy| \le 1$

$|xy| \le 1 \implies xy = z$, where$|z| = 0$ or 1

Thus $z \in P^*$

Hence we have $(x, y) \implies xy \in P^*$

P^*satisfies the following axioms for a pergroup.

P1. $|x.1| = |x.1| < 1, \forall\, x \in P^*$and

$x.1 = 1.x = x$, for $x \in G$

P2. $|xx^{-1}| = |x^{-1}x| = 0, \forall\, x \in P^*$, *so* xx^{-1} *and* $x^{-1}x$ are both defined more over $xx^{-1} = x^{-1}x = 1$ for $x \in G$.

P3. Suppose xy is defined, then $|xy| \le 1$

By A2 $|(xy)^{-1}| \le 1$ then $(xy)^{-1} \in P^*$, and

$(xy)^{-1} = y^{-1}x^{-1}$ for $x, y \in G$

Hence $y^{-1}x^{-1}$ is defined

P4. Suppose $, y, z \in P^*$, such that xy and yz are defined, and suppose $x\,(yz)$ is also defined.

Then $|x(yz)| = |(xy)z| \leq 1$, for $x, y, z \in G$

Hence $(xy)z \in P^*$, and

$x(yz) = (xy)z$

P5. Let $w, x, y, z \in P^*$ m with wx, xy, yz all defined, i.e.

$|wx|, |xy|, |yz| \leq 1$

If one of $|w|$ or $|xy|$or $|z| = 0$, then at least one of $w\,(xy)$ and $(xy)z$ is defined

Suppose neither $|w|$ nor $|xy|$ nor $|z|$ is zero, and suppose neither $w\,(xy)$ nor $(xy)z$ is defined .

i.e. $|w(xy)| = |(xy)z| = 2$

$|w.(xy).z| \geq 3$in G , by applying proposition 3 (page 2) on w,xy, z.

$|(wx).(yz)| \leq 2$in G, but $w.(xy).z = (wx).(yz)$ in G, which is a contradiction.

Therefore, either $w\,(xy)$ is defined or $(xy).z$ is defined, and hence P5 is satisfied [3].

P6. Suppose (x_1, x_2) is a reduced sequence, for x_1, x_2in P^*, then $|x_1| = |x_2| = 1$ and $d\,(x_1, x_2^{-1}) = 0$

Suppose a is an element in P^*such that $x_1 a$ and $a^{-1}x_2$ are both defined then

$2d\,(x_1, a^{-1}) = |x_1| + |a| - |x_1 a| \geq |a|$

i.e. $d\,(x_1, a^{-1}) \geq \frac{1}{2}|a|$.

Also $d\,(a^{-1}, x_2^{-1}) \geq \frac{1}{2}|a|$

Then by A4, $|a| = 0$, $i.e. a \in A_0$.

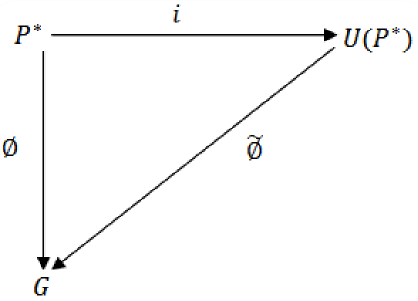
Therefore , if G is a group with length function such that G is generated by element of length zero and one , then these elements from a pregroup satisfying P6 , where the product xy of two elements x and y of $P^*$is defined if and only if $|xy| \leq 1$.

Theorem 3.1: If G is a group with length function such that G is generated by elements of length zero and one, and $P^*$is the pregroup consisting of these elements, then $\cong U\,(P^*)$, This was also observed in [3].

Proof G is generated by elements of P^*, so let $\emptyset : P^* \to G$ be the inclusion.

Then by [3], there exists a specific morphism: $P^* \to U(P^*)$, such that , for any group G and any mprphim $\emptyset : P^* \to G$, there exist unique homomorphism $\tilde{\emptyset} : U(P^*) \to G$ such that $\emptyset = \tilde{\emptyset}\, i$.

Figure 1: Unique homomorphism diagram



Since $\emptyset$ (P^*) generates G , then $\emptyset$ is onto.

Suppose $w = x_1 , \ldots , x_n$ is a word in $U(P^*)$ for minimum n such that

$$\widetilde{\emptyset}(w) = e \in G , \quad \widetilde{\emptyset}(w) = \emptyset (x_1 , x_2 , \ldots , x_n) = x_1 , \ldots , x_n = e \; in \; G$$

When n=1 ,$\emptyset(x_1) = x_1 = e \in G$, then $x_1 = 1 \in U(P^*)$.

Suppose n>1 , then $x_i \, x_{i+1} = x_j$ for same i , $1 \leq i \leq n-1$ where $|x_j| = 0{,}1$; otherewise $|x_i \, x_{i+1}| = 2$, i.e. $d\,(x_i , x_{i+1}) = 0$, I =1 , …., n-1 in which case by [3], $|x_1 \ldots x_n| = n$ and so $x_1 \ldots x_n \neq e$ contradicting $\widetilde{\emptyset}(w) = e$.

Thus w contains a subword of the form $x_i \, x_{i+1}$ where $|x_i \, x_{i+1}| = 0{,}1$ and $x_i \, x_{i+1} = x_j$

i.e. n is not the minimum, which is a contradiction. Hence n = 1 .

Therefore w = e.So $\emptyset$ is an isomorphim.

4. Conclusion

We have now concluded that there are two new stronger versions of P6. These are P6(1) and P6(2) which are equivalent to each other. The equivalent version is denoted by P6* and those pregroups satisfying this condition is dented by P*. Considering the universal group of P* in the normal way we get U(P*) which we have proved that this group satisfies length function defined by Lyndon.

The final conclusion is that the universal group U(P*) is equivalent to G where G is a group generated by element of length zero and one as defined in [3].

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