*Available online www.jsaer.com*

*Journal of Scientific and Engineering Research*, 2017, 4(8):365-368

**Research Article**

**ISSN: 2394-2630**
**CODEN(USA): JSERBR**

# On General Construction of d-dimensional Linear Spaces

**Hasan Keleş**

Karadeniz Technical University, Faculty of Science, Department of Mathematics, 61080, Ortahisar, Trabzon, Turkey

**Abstract** As we known that the notion of linear space can be given in tern the lines of the set of points, which satisfying certain axioms. In this case all the linear spaces in the usual sense, in particular the Euclidean plane (spaces), are a linear space in the sense *line* and *points*. In this work we want begin and investigate the generalization of the notion of linear space in tern the fuzzy lines.

Let $\mathbb{P}$ be a finite set with at least three points, $L$ be a finite $F$ -lattice (i.e. completely distributive lattice with an order-reversing involution $':L \to L$ ) and $\mathbb{L} \subseteq L^{\mathbb{P}}$ be a set of the all Fuzzy subsets of $\mathbb{P}$.

**MSC2000:** 06F20, 46A16, 47S50, 54F05.

**Keywords** d-dimensional fuzzy linear spaces, linear spaces, fuzzy near-linear spaces

## Introduction

Let's start with the following definition, which brings a different view to d-dimensional linear spaces.

**Definition 1.** *For a subfamily (so called fuzzy lines)* $\mathbb{L}$ *subset* $L^{\mathbb{P}}$ *and* $\mathbb{P}$ *points the pair* $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ is called *fuzzy near-linear spaces* $(FNLS)$ *if it satisfies the following condition.*

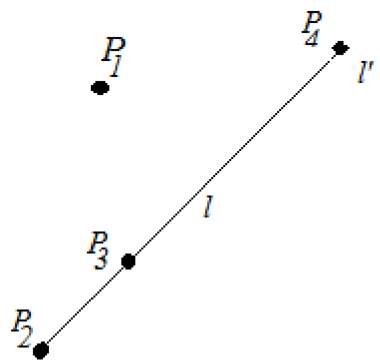
$(FLS-1)$ *For any two different points* $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{P}$ *and* $l \in \mathbb{L}$ *there exist such that* $l(P) \wedge l(Q) \neq 0$.

The two different points in $\mathbb{P}$ are on a line of $\mathbb{L}$.

It is easy to see that if lattice $L$ is trivial (i.e. $L = \{0,1\}$ ), we obtain usual traditional *near-linear space*.

Let us to consider the following example.

**Example 1.** Let $\mathbb{P} = \{P_1, P_2, P_3, P_4\}$, $L = \{0, a, 1\}$ and $\mathbb{L} = \{l, l'\}$ with $l \equiv (1,0,1,a), l' \equiv (1,0,a,1)$.




It is obvious that $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ is a fuzzy near-linear space. Its we briefly denote as

$$\mathbb{S} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & a \\ a & 1 \end{bmatrix}$$

Similarly, for $\mathbb{P}$ and L the following matrix also determines a fuzzy near-linear space:

$$\mathbb{S}_1 = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & a \\ 1 & a & 0 \\ a & a & 1 \end{bmatrix}, \mathbb{S}_2 = \begin{bmatrix} 0 & a & a & 0 \\ a & 0 & a & a \\ 1 & a & a & 1 \\ 1 & 0 & a & a \end{bmatrix}, \mathbb{S}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ a & a & a & 1 \\ 1 & 1 & a & 1 \end{bmatrix}, \ldots$$

**Definition 2.** Let the pair $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ be a fuzzy near-linear spaces. $\mathbb{S}$ is called (fuzzy linear spaces) (FLS) if it satisfies the following condition:

$(FLS-2)$ For any $l \in \mathbb{L}$ there exist $\exists \boldsymbol{P}, \boldsymbol{Q} \in \mathbb{P}$ such that $l(P) \wedge l(Q) \neq 0$.

Any $l$ line of L is at least two points.

$(FLS-2)$ For any $l \in \mathbb{L}$ there exist $\exists \boldsymbol{P}, \boldsymbol{Q} \in \mathbb{P}$ such that $l(P) \wedge l(Q) \neq 0$.

In general speaking, in the definition of the FLS should to require also:

(FLS-3) There exist $\exists \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathbb{P}$ such that, for all $l \in \mathbb{L}$ holds

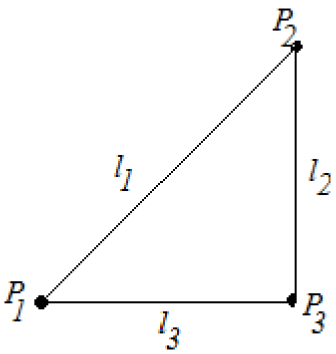
$$l(P) \wedge l(Q) \wedge l(R) = 0.$$

There are three points, three of which are not on the same line.

Throughout in present paper we consider FLS with condition (FLS-3).

**Example 2.** Let $\mathbb{P} = \{P_1, P_2, P_3\}$ and $L = \{0,1\}$ there exists unique fuzzy linear spaces.

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$




If $L_1 = \{0, a, 1\}$ then we can have at most $64$ fuzzy linear spaces.

Checked directly that for the $L_n = \{0, a_1, a_2, \ldots, a_n, 1\}$ finite F-lattice and $|\mathbb{L}| = 3$ general number of the fuzzy linear spaces is[1]:

[1] For the *order*, or number of elements a set $X$, we use $|X|$

$$|\mathbb{S}| = \frac{1}{n!}\sum_{i_1=0}^{6}\sum_{i_2=0}^{6-i_1}\cdots\sum_{i_n}^{6-i_1-\cdots-i_{n-1}}\sum_{i_1+\ldots+i_n<6}\prod_{k=1}^{n}C_{6-i_1-\cdots-i_{n-1}}^{i_k} = (1+n)^6$$

In generally, a straightforward calculation shows that is true the following theorem:

**Theorem 1.** *Let $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ be a FNLS and $V_j =: |\{P \in \mathbb{P}, l_j(P) \neq 0\}|$ for $l_j \in \mathbb{L}$.*
*Then*

$$|\mathbb{S}| = \frac{1}{n!}\sum_{i_1=0}^{|D|}\sum_{i_2=0}^{|D|-i_1}\cdots\sum_{i_n}^{|D|-i_1-\cdots-i_{n-1}}\sum_{i_1+\ldots+i_n<|P|}\prod_{k=1}^{n}C_{|P|-i_1-\cdots-i_{n-1}}^{i_k} = \prod_{i=1}^{|P|}(1+n)^{v_j}$$

Let $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ be a FNLS. For $P, P_j \in \mathbb{P}$ and $l, l_j \in \mathbb{L}$ we introduce the following notations:

$$v(l) =: |\{P \in \mathbb{P}, l(P) \neq 0\}|, b(P) := |\{l \in \mathbb{L}, l(P) \neq 0\}|$$

$$C_{ij} =: |\{Q \in \mathbb{P}, l_j(Q) \neq 0\}| \text{ and } \exists l \in \mathbb{L}: l(Q) \wedge l_j(P_j) \neq 0,$$

$$r_{ij} = \begin{cases} 0, \text{ if } l_j(P_i) = 0 \\ 1, \text{ if } l_j(P_i) \neq 0 \end{cases}$$

**Proposition 2.** *Let $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ be a FNLS. If $c_{ij} = v(l_j)$ for every $P_i$ and $l_j$ with $r_{ij} = 0$, then $\mathbb{S}$ is a FLS.*

*Proof.* Let $P_i$ and $Q_i$ are the arbitrary two points of $\mathbb{P}$. Since $\mathbb{L} \neq \varnothing$, we can take a line from $\mathbb{L}$, say $l_k$.

If $r_{ik} = r_{jk} = 1$, then $l_k(P_i) \neq 0$ and $l_k(Q_i) \neq 0$. Hence, $l_k(P_i) \wedge l_k(Q_i) \neq 0$, therefore $(FLS-2)$ carry out.

If $r_{ik} = 0$ and $r_{jk} = 0$, then $l_k(P_i) = 0$ and $l_k(Q_i) \neq 0$. By hypothesis there exists $l \in \mathbb{L}$ with $l(P_i) \wedge l(Q_i) \neq 0$, since $c_{ik} = v(l_k)$. Therefore $(FLS-2)$ carry out.

If $r_{ik} = r_{jk} = 0$, then $l_k(P_i) = l_k(Q_i) = 0$. By $(FLS-1)$ the line $l_k$ has at least two points, say one wich $P$ $l_k(P) \neq 0$. By hypothesis there exists $l \in \mathbb{L}$ with $l(P_i) \wedge l(Q_i) \neq 0$, since $c_{ik} = v(l_k)$. If $l(Q_j) \neq 0$, then $(FLS-2)$ carry out. Otherwise (i.e. in the case $l(Q_j) \neq 0$) again by hypothesis there exists $l' \in \mathbb{L}$ such that $l'(P_i) \wedge l'(Q_i) \neq 0$, since $c(Q_j, l) = v(l)$. Therefore $(FLS-2)$ carry out. Thus $\mathbb{S}$ *is a FLS.*

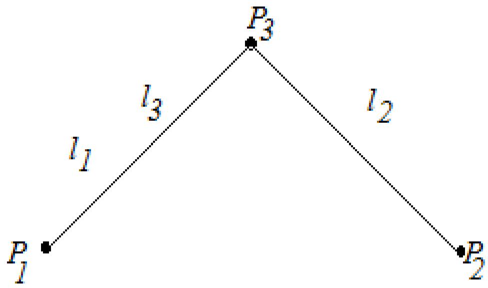
**Theorem 3.** *Let $v := |\mathbb{P}|$ and $b := |\mathbb{L}|$. If $\mathbb{S} = (\mathbb{P}, \mathbb{L}$ is a FLS, then*

$$\sum_{j=1}^{b} v_j(v_j - 1) \geq v(v-1).$$ (*A sum with no entries is assumed to be zero.*)

*Proof.* For a set $\mathbb{P}$, there are $C_v^2 = \frac{v(v-1)}{2}$ pairs of points (counting $\{P_i, Q_j\}$ to be the same pair as $\{P_i, Q_j\}$). Also as any pair of points determines at least one line, therefore, the total number of pairs of points is no more than the total number of pairs of points on each line, i.e. $\frac{v(v-1)}{2} \leq \sum_{j=1}^{b}\frac{v_j(v_j-1)}{2}$.

Unlike traditional linear space, it is not hard to see that here the conversely assertion, general speaking, is not true, as show the following example.

**Example 3.** For $\mathbb{P} = \{P_1, P_2, P_3\}$, $\mathbb{L} = \{l_1, l_2, l_3\}$and $L = \{0, a, 1\}$ we consider $\mathbb{S} = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & a \\ 0 & 1 & 0 \end{bmatrix}$:




Checked directly that $v_1 = v_2 = v_3$ and $\sum_{j=1}^{3} v_j (v_j - 1) = v(v-1) = 6$. But, it is easy to see that $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ isn't a fuzzy linear space.

In order to shown a conversely assertion, we give the following definition.

**Definition 3.** *Let $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ be a FNLS. We say that two lines $l_i$ and $l_j$* are equivalent and denote as $l_i \sim l_j$, *if there exists the points* ***P*** *,**Q*** $\in \mathbb{P}$ *such that* $l_i(P) \wedge l_i(Q) \neq 0$ *and* $l_j(P) \wedge l_j(Q) \neq 0$. *Let* $\mathbb{L}' = \mathbb{L}|_{\sim}$ *be the set of the equivalence classes of* $\mathbb{L}$, *and let*$b' := |\mathbb{L}'|$.

It is easy to prove the following theorem.

**Theorem 4.** $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ *is a FLS iff* $\sum_{j=1}^{b'} v_j (v_j - 1) \geq v(v-1)$.

From Theorem 3 and 4 we obtain:

**Corollary 5.** *If* $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ *is a FLS,* $|\mathbb{L}| \geq 3$ *and there exists* ***P*** *,**Q*** $\in \mathbb{P}$ *with* $\left|\{ l \in \mathrm{L} : l(P) \wedge l(Q) \neq 0 \}\right| \geq 2$, *then* $\sum_{j=1}^{b} v_j (v_j - 1) \geq v(v-1)$.

**Corollary.** *If* $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ *is a FLS,* $L = \{0, 1\}$, then

$$\sum_{j=1}^{b} v_j (v_j - 1) = v(v-1).$$

## References


[1]. Keleş, H., (1996). d-dimensional linear spaces, University of Sakarya, Thesis, Sakarya.

[2]. S. Barwick. (1994). Subsrtuctures of finite geometries, The University of London, Page No: 113.

[3]. L.M. Baten, A. Beutelspacher. (1993). The theory of finite linear spaces. Combinatorics of Points and Lines, Cambrige University Press, Page No: 214.

[4]. L. Batten. (1986). Combinatorics of finite geometries, Cambridge University Press, Page No: 172.

[5]. Koen Thas. (2002). Automorphisms and combinatorics of finite generalized quadrancles, Page No: 440.

**[6].** Keleş, H. (2017). On some properties combinatorics of Graphs in the d-dimensional FLS, Journal of Scientific and Engineering Research, 2017, 4(2):93-97.