Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2017, 4(7):299-308**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Bayesian and E-Bayesian estimations for the compound Rayleigh distribution based on upper record values

A. I. Shawky*, F.M. Al-Aboud

Department of Statistics, Faculty of Science, King Abdulaziz University, PO Box 80203, Jeddah 21589, Saudi Arabia

Abstract The present paper is concerned with using E-Bayesian method under record values sample from compound Rayleigh distribution (CRD) to find estimates for the parameter and hazard function. The Bayesian and E-Bayesian estimators are obtained under different loss functions (squared error, LINEX and entropy loss functions). Relations between E-Bayesian estimations for the parameter and hazard function are discussed. A comparison between E-Bayesian, and corresponding Bayes are investigated by using simulation data and real life data.

Keywords Compound Rayleigh distribution, Bayesian estimation, E- Bayesian estimation, Loss functions, Record values

1. Introduction

The compound Rayleigh distribution (CRD) is one of models which is useful in different areas of statistics, the probability density function (pdf) given by

$$f(x;a,b) = 2\,a\,b^{a}x(b+x^{2})^{-(a+1)},\quad x>0,\quad a,b>0, \tag{1}$$

and the cumulative distribution function (cdf) of CRD given by

$$F(x;a,b) = 1 - b^{a}(b+x^{2})^{-a}\ , \tag{2}$$

where a and b defined as shape and scale parameters, respectively. The survival (reliability) and hazard (failure rate) functions, at time t, of the CRD, respectively, are

$$R(t;a,b) = b^{a}(b+t^{2})^{-a}\ , \tag{3}$$

$$h(t;a,b) = \frac{2\,a\,t}{b+t^{2}},\quad t>0. \tag{4}$$

The CRD is a special case of the 3- parameter Burr type XII distribution, which has a *pdf* of the form (1). Applications of randomly censored data as goodness of fit of the CRD using a medical data are studied by Bekker et al. [8] and Ghitany [14]. For more details properties on CRD(a,b), see El-Sagheer and Ahsanullah [13].

Record values and the associated statistics are of interest and importance in many areas of real life applications involving data relating to sport, economics, athletic events, oil, mining surveys and life testing. Many authors have studied records and associated statistics. Among them are Ahsanullah [3, 4], Resnick [28], Nagaraja [22], Arnold et al. [5], Ragab [27], Soliman and Al-Aboud [34], AbdEllah[1, 2] and El-Sagheer [12].

Let $\{X_m,\ m\geq 1\}$ be an infinite sequence of independent and identically distributed (iid) random variables with cdf$F$(x)andpdf$f$(x). Set $Y_n = \max(X_1,\ X_2,\ldots,X_m)\,, m\geq 1$, we say that X_j is an upper record and denoted by $X_{U(j)}$ if $Y_j > Y_{j-1},\ j>1$.

This paper discusses the record values data from CRD and the maximum likelihood estimation (MLE) to estimate the parameter reliability, and hazard functions in Section 2. Bayesian estimators are derived based on squared error, LINEX and entropy loss functions in Section 3. The E-Bayesian estimates for three different prior distributions of the hyperparameters α and β are discussed in Section 4. Properties of E-Bayesian estimation are carried out in Section 5. While comparisons between the new method, the corresponding Bayes, and MLE`s are studied through numerical examples (simulation data and real life data) in Section 6. Finally, we conclude with some comments in the last Section.

2. Maximum Likelihood Estimation

In general, the joint pdf of the first m upper record values $X_{U(1)}, X_{U(2)}, \dots, X_{U(m)}$ is given by

$$f_{1,2,\dots,m}\left(x_{u(1)}, x_{u(2)}, \dots, x_{u(m)}\right) = f\left(x_{u(m)}\right).\prod_{i=1}^{m} \frac{f(x_{u(i)})}{R(x_{(i)})},$$

$$0 < x_{u(1)} < x_{u(2)} < \dots < x_{u(m)} < \infty. \quad (5)$$

By using (1), (2) and (5) after replacing $x_{u(i)}$ by x_i, therefore without the additive constant, the likelihood function, when b is known, can be written as

$$\ell = a^m . e^{-aT}, \quad (6)$$

where

$$T = log\left(\frac{b+x_m^2}{b}\right). \quad (7)$$

Then, the log-likelihood function is

$$L = m \log(a) - a\,T. \quad (8)$$

Equating the first partial derivative of (8), with respect to a, to zero, we obtain the MLE of a as

$$\hat{a}_{ML} = \frac{m}{T}. \quad (9)$$

Therefore, the MLE of $R(t)$and $h(t)$, respectively, are

$$\hat{R}_{ML}(t) = b^{\hat{a}_{ML}}(b+t^2)^{-\hat{a}_{ML}}, \hat{h}_{ML}(t) = \frac{2\,t\,\hat{a}_{ML}}{b+t^2}. \quad (10)$$

3. Bayes estimation

In this section, we present the posterior densities of the parameter$a$ when b is known and hence derive symmetric and asymmetric Bayes estimators for $a$andhazardfunction.

3.1. The Loss Function

The symmetric square-error loss (SEL) function is one of the most popular loss functions. A useful asymmetric loss function known as the LINEX loss function was introduced by [35] and was widely used in several papers, see for example, [7- 9, 21, 26, 32, 33, 35].

The Bayes estimator $\hat{a}_{BL}$ of a under the LINEX loss function is

$$\hat{a}_{BL} = \frac{-1}{s}\log[E(e^{-sa})], s \neq 0, \quad (11)$$

provided that the expectation $E(e^{-sa})$ exists and is finite.

An other useful asymmetric loss function is the entropy loss (BE), see [10] and [11],

and the estimator of a under BE function is

$$\hat{a}_{BE} = [E(a^{-1})]^{-1}, \quad (12)$$

provided that the expectation $E(a^{-1})$ exists and finite.

3.2. Prior and posterior distributions

Assuming the parameter b is known, we can use the gamma distribution as prior distribution of a with shape and scale parameters α and β respectively and its pdf given by

$$g(a\,|\,\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} a^{\alpha-1} e^{-\beta a}, \quad a > 0, \ \alpha,\beta > 0. \quad (13)$$

Combining (6) and (13), we get the posterior density of a given $\underline{x}$can be obtained as

$$\Pi(a|\underline{x}) = \frac{(T+\beta)^{m+\alpha}}{\Gamma(m+\alpha)} a^{m+\alpha-1} e^{-(T+\beta)a}, \quad (14)$$

which obeys the gamma distribution with parameters $m+\alpha$and $T+\beta$.

3.3. Bayesian estimation under squared error loss (SEL) function

Under the squared error loss function, the Bayes estimates of a,$R(t)$and $h(t)$ are, respectively, given by

$\hat{a}_{BS} = E(a) = \frac{m+\alpha}{T+\beta}$, $\hat{R}_{BS} = (\frac{T+\beta}{T+T1+\beta})^{m+\alpha}, \hat{h}_{BS} = \frac{2(m+\alpha)t}{(b+t^2)(T+\beta)}$, (15)

where

$T1 = \ln\left(1+\frac{t^2}{b}\right)$. (16)

3.4. Bayesian estimation under LINEX loss function

By using (11) and (14), then the Bayes estimates of θ, $R(t)$ and $h(t)$under LINEX loss function are, respectively, given by

$\hat{a}_{BL} = \frac{m+\alpha}{S}\log(1+\frac{S}{T+\beta})$, $\hat{R}_{BL} = \frac{-1}{S}\log[\sum_{i=0}^{\infty}\frac{(-S)^i}{i!}(\frac{T+\beta}{T+i\,T1+\beta})^{m+\alpha}]$,

$\hat{h}_{BL} = \frac{m+\alpha}{s}\log(1+\frac{2\,s\,t}{(b+t^2)(T+\beta)})$. (17)

3.5. Bayesian estimation under Entropy oss function

By using (12) and (14), then the Bayes estimates of θ, $R(t)$ and $h(t)$under entropy loss function are, respectively, given by

$\hat{a}_{BE} = \frac{m+\alpha-1}{T+\beta}$, $\hat{R}_{BE} = (1-\frac{T1}{T+\beta})^{m+\alpha}\hat{h}_{BE} = \frac{2\,t\,(m+\alpha-1)}{(b+t^2)(T+\beta)}$. (18)

4. E-Bayesian Estimation

According to Han [16], the prior parameters$\alpha$and β must be choose to guarantee that $g(a)$ is a decreasing function of a. The derivative of $g(a|\alpha,\beta)$ with respect to a is

$\frac{dg(a|\alpha,\beta)}{da} = \frac{\beta^{\alpha}}{\Gamma(\alpha)}a^{\alpha-2}e^{-\beta a}[(\alpha-1)-\beta\,a]$, (19)

where$a>0$, $\alpha,\beta>0$, it follows $0<\alpha<1$, $\beta>0$ due to $\frac{dg(a|\alpha,\beta)}{da}<0$, and therefore $g(a|\alpha,\beta)$ is a decreasing function of a.

Suppose that α and β are independent with bivariate density function

$\Pi(\alpha,\beta) = \Pi_1(\alpha)\Pi_2(\beta)$. (20)

Then, the E-Bayesian estimate of a (expectation of the Bayesian estimate of a) can be written as

$\hat{a}_{EB} = E(a|\underline{x}) = \iint_Q \hat{a}_B(\alpha,\beta)\Pi(\alpha,\beta)d\alpha d\beta$, (21)

where Q is the domain of α and β, and $\hat{a}_B(\alpha,\beta)$ is the Bayes estimate of a given by (15), (17) and (18). For more details, see Han [17, 18], Azimi et al. [6], Gupta and Gupta [15], Kizilaslan [19], Li et al. [20], Nasiri and Esfandyarifar [23], Okasha [24], Okasha and Wang [25], Reyad and Ahmed [29, 30], Read et al,.[31], Wang et al. [36] and Yousefzadeh [37].

The following joint prior density functions of α,β, where $0<\alpha<1, 0<\beta<c$, may be used

$\Pi_1(\alpha,\beta) = \frac{2(c-\beta)}{c^2}$, $0<\alpha<1$, $0<\beta<c$,

$\Pi_2(\alpha,\beta) = \frac{1}{c}$, $0<\alpha<1$, $0<\beta<c$,

$\Pi_3(\alpha,\beta) = \frac{2\beta}{c^2}$, $0<\alpha<1$, $0<\beta<c$. (22)

4.1. E-Bayesian estimation under squared error loss function

For $\pi_i(\alpha,\beta), i=1,2,3,$ the E-Bayesian estimate of a and $h(t)$based on the squared error loss function, are derived from (15) and (22)as

$$\hat{a}_{EBS1} = \iint_Q \hat{a}_{BS}(\alpha,\beta)\Pi_1(\alpha,\beta)d\alpha d\beta = \int_0^c\int_0^1 \frac{2(c-\beta)}{c^2}.\frac{m+\alpha}{T+\beta}d\alpha d\beta$$

$$= \frac{2m+1}{c^2}\left\{(T+c)\log\left(\frac{T+c}{T}\right)-c\right\},$$

$$\hat{a}_{EBS2} = \frac{2m+1}{2\,c}\log\left(\frac{T+c}{T}\right), \hat{a}_{EBS3} = \frac{2m+1}{c^2}\{c-T\log\left(\frac{T+c}{T}\right)\},$$

$$\hat{h}_{EBS1} = \frac{2\,t\,(2m+1)}{c^2(b+t^2)}\left\{(T+c)\log\left(\frac{T+c}{T}\right)-c\right\}, \hat{h}_{EBS2} = \frac{t\,(2m+1)}{c(b+t^2)}\log\left(\frac{T+c}{T}\right),$$

$$\hat{h}_{EBS3} = \frac{2\,t\,(2m+1)}{c^2(b+t^2)}\left\{c - T\log\left(\frac{T+c}{T}\right)\right\}. \quad (23)$$

4.2. E-Bayesian estimation under LINEX loss function

For $\pi_i(\alpha,\beta), i = 1, 2, 3,$ the E-Bayesian estimate of a and $h(t)$based on the LINEX loss function, are derived from (17) and (22) as

$$\hat{a}_{EBL1} = \iint_Q \hat{a}_{BL}(\alpha,\beta)\Pi_1(\alpha,\beta)d\alpha d\beta = \int_0^c\int_0^1 \frac{2(c-\beta)}{c^2}.\frac{m+\alpha}{S}\log(1+\frac{S}{T+\beta})d\alpha d\beta$$

$$= \frac{2m+1}{c^2 S}(c\,I1 - J1),$$

$$\hat{a}_{EBL2} = \frac{2m+1}{2cS}I1, \hat{a}_{EBL3} = \frac{2m+1}{c^2 S}J1,$$

$$\hat{h}_{EBL1} = \frac{2m+1}{c^2 S}(c\,I2 - J2), \qquad \hat{h}_{EBL2} = \frac{2m+1}{2c\,S}I2, \hat{h}_{EBL3} = \frac{2m+1}{c^2 S}J2, (24)$$

where

$$I1 = c\log\left(1+\frac{S}{T+c}\right) - T\log\left(1+\frac{c}{T}\right) + (T+S)\log\left(1+\frac{c}{T+S}\right),$$

$$J1 = \frac{1}{2}\{c^2\log\left(1+\frac{S}{T+c}\right) + cS + T^2\log\left(1+\frac{c}{T}\right) - (T+S)^2\log\left(1+\frac{c}{T+S}\right)\},$$

$$I2 = c\log\left(1+\frac{H}{T+c}\right) - T\log\left(1+\frac{c}{T}\right) + (T+H)\log\left(1+\frac{c}{T+H}\right),$$

$$J2 = \frac{1}{2}\{c^2\log\left(1+\frac{H}{T+c}\right) + cH + T^2\log\left(1+\frac{c}{T}\right) - (T+H)^2\log\left(1+\frac{c}{T+H}\right)\},$$

$$H = \frac{2t\,S}{b+t^2}. \quad (25)$$

4.3. E-Bayesian estimation under Entropy loss function

For $\pi_i(\alpha,\beta), i = 1, 2, 3,$ the E-Bayesian estimate of a and $h(t)$based on the entropy loss function, are derived from (18) and (22) as

$$\hat{a}_{EBE1} = \frac{2m-1}{c^2}\left\{(T+c)\log\left(\frac{T+c}{T}\right) - c\right\}, \hat{a}_{EBE2} = \frac{2m-1}{2c}\log\left(\frac{T+c}{T}\right),$$

$$\hat{a}_{EBE3} = \frac{2m-1}{c^2}\left\{c - T\log\left(\frac{T+c}{T}\right)\right\}, \hat{h}_{EBE1} = \frac{2\,t\,(2m-1)}{c^2(b+t^2)}\left\{(T+c)\log\left(\frac{T+c}{T}\right) - c\right\},$$

$$\hat{h}_{EBE2} = \frac{t\,(2m-1)}{c(b+t^2)}\log\left(\frac{T+c}{T}\right), \hat{h}_{EBE3} = \frac{2\,t\,(2m-1)}{c^2(b+t^2)}\left\{c - T\log\left(\frac{T+c}{T}\right)\right\}. \quad (26)$$

5. Properties of E-Bayesian estimation

In this section, the relations between the E-Bayesian estimators $\hat{a}_{EBSi}$, $\hat{a}_{EBLi}$,$\hat{a}_{EBEi}$$\hat{h}_{EBSi}$, $\hat{h}_{EBLi}$ and $\hat{h}_{EBEi}$, $i = 1, 2, 3$ are discussed.

5.1. Relations among $\hat{a}_{EBSi}$, $\hat{a}_{EBLi}$ and $\hat{a}_{EBEi}$(i=1, 2, 3)

Lemma 1. Let $0 < c < T$ and $\hat{a}_{EBSi}$, $\hat{a}_{EBLi}$ and $\hat{a}_{EBEi}$ (i=1, 2, 3) be given by Eqs (23), (24) and (26). Then

i) $\hat{a}_{EBS1} > \hat{a}_{EBS2} > \hat{a}_{EBS3}$,

ii) $\lim_{T\to\infty}\hat{a}_{EBS1} = \lim_{T\to\infty}\hat{a}_{EBS2} = \lim_{T\to\infty}\hat{a}_{EBS3}$,

iii)$\hat{a}_{EBL1} > \hat{a}_{EBL2} > \hat{a}_{EBL3}$,

iv)$\lim_{T\to\infty}\hat{a}_{EBL1} = \lim_{T\to\infty}\hat{a}_{EBL2} = \lim_{T\to\infty}\hat{a}_{EBL3}$,

v) $\hat{a}_{EBE1} > \hat{a}_{EBE2} > \hat{a}_{EBE3}$,

vi)$\lim_{T\to\infty}\hat{a}_{EBE1} = \lim_{T\to\infty}\hat{a}_{EBE2} = \lim_{T\to\infty}\hat{a}_{EBE3}$.

Proof. See Appendix.

5.2. Relations among $\hat{h}_{EBSi}$, $\hat{h}_{EBLi}$ and $\hat{h}_{EBEi}(i = 1, 2, 3)$

Lemma 1. Let $0 < c < T$ and $\hat{h}_{EBSi}$, $\hat{h}_{EBLi}$ and $\hat{h}_{EBEi}$ (i=1, 2, 3) be given by Eqs (23), (24) and (26). Then

i) $\hat{h}_{EBS1} > \hat{h}_{EBS2} > \hat{h}_{EBS3}$,

ii) $\lim_{T\to\infty}\hat{h}_{EBS1} = \lim_{T\to\infty}\hat{h}_{EBS2} = \lim_{T\to\infty}\hat{h}_{EBS3}$,

iii) $\hat{h}_{EBL1} > \hat{h}_{EBL2} > \hat{h}_{EBL3}$,

iv)$\lim_{T\to\infty}\hat{h}_{EBL1} = \lim_{T\to\infty}\hat{h}_{EBL2} = \lim_{T\to\infty}\hat{h}_{EBL3}$,

v) $\hat{h}_{EBE1} > \hat{h}_{EBE2} > \hat{h}_{EBE3}$,
vi)$\lim_{T\to\infty} \hat{h}_{EBE1} = \lim_{T\to\infty} \hat{h}_{EBE2} = \lim_{T\to\infty} \hat{h}_{EBE3}$.
Proof. See Appendix.

6. Simulation study and Application example

In this section, two examples are presented to illustrate the use of E-Bayesian in estimating the parameter and hazard function.

Example 1 (Simulated data)

Comparisons of the new method (E-Bayesian estimation) and Bayesian estimationof the parameterand hazard function are discussed by simulation study. The following steps are
i) For given value of the prior parameter c, α and β were generated from uniform priors (22), respectively.
ii) For given values of $\alpha$and β, a was generated from the gamma prior density (13).
iii) For known value of $b$and the value of a, from the above step, we generate samples of record values with different sizes (m=3, 5, 7) from CRD with pdf (1).
iv)Under the different loss functions, the estimates $\hat{a}_{BS}$,$\hat{a}_{EBSi}$,$\hat{a}_{EBLi}$ $\hat{a}_{EBEi}$, $\hat{h}_{BS}$, $\hat{h}_{EBSi}$,$\hat{h}_{EBLi}$ and $\hat{h}_{EBEi}$ $(i = 1, 2, 3)$ were computed from (15), (17), (18), (23), (24 and (26).
v) We repeat the above steps 1000 times and then we compute the mean square errors (MSEs) and average for the estimates. The simulation results are displayed in Tables 1-2.

Table 1: MSEs and average estimates for a

n	$\hat{a}_{ML}$	$\hat{a}_{BS}$	$\hat{a}_{EBS}$	$\hat{a}_{BL}$	$\hat{a}_{EBL}$	$\hat{a}_{BE}$	$\hat{a}_{EBE}$
3	0.0855145 (0.207981)	0.0441244 (0.238676)	0.0434731 (0.222102)	0.0299992 (0.222508)	0.029772 (0.204952)	0.0250197 (0.17849)	0.0240636 (0.158644)
			0.0386747 (0.2141)		0.0253397 (0.198755)		0.0227437 (0.152928)
			0.030375 (0.206098)		0.0214378 (0.192559)		0.0191885 (0.147213)
5	0.0429353 (0.216043)	0.0224504 (0.241825)	0.0213458 (0.228482)	0.0181962 (0.232448)	0.0171324 (0.219224)	0.0153118 (0.201289)	0.0150327 (0.18694)
			0.0193065 (0.224271)		0.0157258 (0.215454)		0.0139383 (0.183494)
			0.0174201 (0.220059)		0.0144183 (0.211685)		0.0129462 (0.180048)
7	0.030508 (0.247617)	0.0198405 (0.268566)	0.019003 (0.257574)	0.0169071 (0.260394)	0.0161795 (0.249505)	0.014999 (0.23485)	0.0147936 (0.223231)
			0.017662 (0.253926)		0.0151684 (0.246137)		0.0139951 (0.220069)
			0.0164016 (0.250278)		0.0142179 (0.242769)		0.0132571 (0.216908)

Table 2: MSEs and average estimates for $h(t)$

n	$\hat{h}_{ML}$	h_{BS}	$\hat{h}_{EBS}$	$\hat{h}_{BL}$	$\hat{h}_{EBL}$	$\hat{h}_{BE}$	$\hat{h}_{EBE}$
3	1.07426 (0.922951)	0.176818 (0.584421)	0.1544041 (0.759233)	0.206289 (0.511616)	0.107574 (0.638322)	0.291527 (0.412532)	0.125832 (0.549452)
			0.137029 (0.681652)		0.06647236 (0.578387)		0.0706191 (0.486894)
			0.0672756 (0.59407)		0.0386046 (0.518452)		0.0544161 (0.424335)

5	0.890664 (1.04907)	0.169715 (0.760319)	0.151584 (0.922752) 0.122204 (0.842369) 0.0561992 (0.761983)	0.19113 (0.681871)	0.10518 (0.804987) 0.0646118 (0.744052) 0.037257 (0.683117)	0.241835 (0.619519)	0.11902 (0.754979) 0.0655444 (0.689211) 0.048421 (0.623444)
7	0.70669 (1.20253)	0.165228 (0.917195)	0.151023 (1.07416) 0.120801 (0.994301) 0.105719 (0.914443)	0.108988 (0.833727)	0.104443 (0.958019) 0.101914 (0.894732) 0.0717875 (0.831445)	0.240404 (0.79325)	0.116229 (0.930937) 0.069501 (0.861727) 0.0579402 (0.792518)

Example 2 (Real life data)

This example provides a real data set to illustrate the estimation methods which have been described in the preceding sections. All the computations were performed using Mathematica version 9. We select seven observations of the real data from Bekker et al. [8] to consider as an upper record values {0.164, 0.501, 0.863, 1.485, 2.178, 2.416, 3.578}. Based on these seven record values and $c = 2, S = 1.5, b = 2.5, t = 1.5, \alpha = 0.9477$ and $\beta = 1.06365$. Therefore, the results are displayed in Table (3)

Table 3: Average values of the different estimators for the parameters a and h.

$\hat{a}_{ML}$	$\hat{a}_{BS}$	$\hat{a}_{EBS}$	$\hat{a}_{BL}$	$\hat{a}_{EBL}$	$\hat{a}_{BE}$	$\hat{a}_{EBE}$
3.86378	2.80441	3.13192 2.78929 2.44665	2.22436	2.41965 2.20182 1.98399	2.41630	2.71433 2.41738 2.12043
$\hat{h}_{ML}$	h_{BS}	$\hat{h}_{EBS}$	$\hat{h}_{BL}$	$\hat{h}_{EBL}$	$\hat{h}_{BE}$	$\hat{h}_{EBE}$
2.44028	1.74574	1.97806 1.76166 1.54525	1.50894	1.66010 1.50131 1.34252	1.52608	1.71432 1.52677 1.33922

7. Concluding Remarks

Based on the results shown in Tables 1-2, one concludes generally, the MSEs of the E-Bayesian estimates of a and h are the smallest MSE compared with their corresponding Bayes estimates. The MSE ofthe E-Bayesian estimates under LINEX loss function have smallest MSE compared with the E-Bayesian estimates under squared error loss function. Also was noted that MSEs of Bayesian and E-Bayesian estimates decrease as n increase. By increasing n, the computations in all tables show that the E-Bayes estimates (based on squared error, LINEX and entropy losses) are better than the Bayes in the sense of comparing the MSEs of the estimates. Table 3 is conformed the properties of E-Bayesian estimation which discussed in Section 5.

Appendix

Proof of Lemma 1

i) From (23), we have

$\hat{a}_{EBS1} - \hat{a}_{EBS2} = \hat{a}_{EBS2} - \hat{a}_{EBS3} = \frac{2m+1}{2\,c^2} M,$ (A.1)

where

$$M = (2T + c)\, log\left(1 + \frac{c}{T}\right) - 2c$$

For -1 < y < 1, we deduce

$$log(1 + y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - \frac{1}{6}y^6 + \cdots = \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{y^\ell}{\ell}.$$

Let $y = \frac{c}{T}$, when $0 < \frac{c}{T} < 1$, we get

$$M = c\left(\frac{2\,T}{c} + 1\right)\left[\frac{c}{T} - \frac{1}{2}\left(\frac{c}{T}\right)^2 + \frac{1}{3}\left(\frac{c}{T}\right)^3 - \frac{1}{4}\left(\frac{c}{T}\right)^4 + \frac{1}{5}\left(\frac{c}{T}\right)^5 - \frac{1}{6}\left(\frac{c}{T}\right)^6 + \frac{1}{7}\left(\frac{c}{T}\right)^7 + \cdots\right] - 2\,c$$

$$= c\left\{\frac{1}{6}\left(\frac{c}{T}\right)^2\left(1 - \frac{c}{T}\right) + \frac{1}{60}\left(\frac{c}{T}\right)^4\left[9 - 8\left(\frac{c}{T}\right)\right] + \cdots\right\} > 0. \text{(A.2)}$$

According to (A.1) and (A.2), we obtain

$\hat{a}_{EBS1} - \hat{a}_{EBS2} = \hat{a}_{EBS2} - \hat{a}_{EBS3} > 0,$

that is

$\hat{a}_{EBS1} > \hat{a}_{EBS2} > \hat{a}_{EBS3}.$

ii) From (A.1) and (A.2), we get

$$\lim_{T\to\infty}(\hat{a}_{EBS1} - \hat{a}_{EBS2}) = \lim_{T\to\infty}(\hat{a}_{EBS2} - \hat{a}_{EBS3}) = \frac{2m+1}{2c^2}\lim_{T\to\infty} M = 0.$$

Then,

$$\lim_{T\to\infty}\hat{a}_{EBS1} = \lim_{T\to\infty}\hat{a}_{EBS2} = \lim_{T\to\infty}\hat{a}_{EBS3}.$$

iii) From (24), we deduce

$\hat{a}_{EBL1} - \hat{a}_{EBL2} = \hat{a}_{EBL2} - \hat{a}_{EBL3} = \frac{2m+1}{2\,S} M1$, (A.3)

Where

$$M1 = \frac{1}{c^2}(cI1 - 2\,J1) = -\frac{S}{c} - \frac{T}{c}\left(1 + \frac{T}{c}\right) log\left(1 + \frac{c}{T}\right) + \left(\frac{T+S}{c}\right)\left(1 + \frac{T+S}{c}\right) log\left(1 + \frac{c}{T+S}\right)$$

$$= \frac{c}{T}\left[\frac{1}{6} - \frac{1}{12}\left(\frac{c}{T}\right) + \frac{1}{20}\left(\frac{c}{T}\right)^2 - \frac{1}{30}\left(\frac{c}{T}\right)^3 + \frac{1}{42}\left(\frac{c}{T}\right)^4 - \frac{1}{56}\left(\frac{c}{T}\right)^5 + \cdots\right]$$

$$-\frac{c}{T+S}\left[\frac{1}{6} - \frac{1}{12}\left(\frac{c}{T+S}\right) + \frac{1}{20}\left(\frac{c}{T+S}\right)^2 - \frac{1}{30}\left(\frac{c}{T+S}\right)^3 + \frac{1}{42}\left(\frac{c}{T+S}\right)^4 - \frac{1}{56}\left(\frac{c}{T+S}\right)^5 + \cdots\right],$$

we note that

$M1 > 0$, if $S > 0$ and $M1 < 0$, if $S < 0$. (A.4)

Therefore, from (A.3) and (A.4), we obtain

$\hat{a}_{EBL1} - \hat{a}_{EBL2} = \hat{a}_{EBL2} - \hat{a}_{EBL3} > 0,$

that is

$\hat{a}_{EBL1} > \hat{a}_{EBL2} > \hat{a}_{EBL3}.$

iv) According to (A.3) and (A.4), we get

$$\lim_{T\to\infty}(\hat{a}_{EBL1} - \hat{a}_{EBL2}) = \lim_{T\to\infty}(\hat{a}_{EBL2} - \hat{a}_{EBL3}) = \frac{2m+1}{2S}\lim_{T\to\infty} M1 = 0.$$

Then,

$$\lim_{T\to\infty}\hat{a}_{EBL1} = \lim_{T\to\infty}\hat{a}_{EBL2} = \lim_{T\to\infty}\hat{a}_{EBL3}.$$

v), vi) From (26), we have

$\hat{a}_{EBE1} - \hat{a}_{EBE2} = \hat{a}_{EBE2} - \hat{a}_{EBE3} = \frac{2m-1}{2\,c^2} M$, (A.5)

therefore, as (i) and (ii), we can prove the relations.

Proof of Lemma 2

i) From (23), we deduce

$\hat{h}_{EBS1} - \hat{h}_{EBS2} = \hat{h}_{EBS2} - \hat{h}_{EBS3} = \frac{(2m+1)t}{c^2(b+t^2)} M$, (A.6)

and from Lemma 1, we have $M > 0$, thus

$\hat{h}_{EBS1} - \hat{h}_{EBS2} = \hat{h}_{EBS2} - h_{EBS3} > 0,$

that is

$\hat{h}_{EBS1} > \hat{h}_{EBS2} > h_{EBS3}.$

ii) According to (A.2) and (A.6), we get

$$\lim_{T\to\infty}\left(\hat{h}_{EBS1} - \hat{h}_{EBS2}\right) = \lim_{T\to\infty}\left(\hat{h}_{EBS2} - h_{EBS3}\right) = \frac{(2m+1)t}{c^2(b+t^2)}\lim_{T\to\infty} M = 0.$$

Then,

$$\lim_{T\to\infty}\hat{h}_{EBS1} = \lim_{T\to\infty}\hat{h}_{EBS2} = \lim_{T\to\infty}\hat{h}_{EBS3}.$$

iii) From (24), we deduce

$\hat{h}_{EBL1} - \hat{h}_{EBL2} = \hat{h}_{EBL2} - \hat{h}_{EBL3} = \frac{2m+1}{2S}M2,$ (A.7)

Where

$$M2 = \frac{1}{c^2}(c\,I2 - 2J2)$$

$$M2 = \frac{T+St}{c}\left(1+\frac{T+St}{c}\right)log(1+\frac{c}{T+St}) - \frac{T}{c}\left(1+\frac{T}{c}\right)\,log\left(1+\frac{c}{T}\right) - \frac{St}{c}$$

$$= \frac{c}{T}\left[\frac{1}{6} - \frac{1}{12}\left(\frac{c}{T}\right) + \frac{1}{20}\left(\frac{c}{T}\right)^2 - \frac{1}{30}\left(\frac{c}{T}\right)^3 + \frac{1}{42}\left(\frac{c}{T}\right)^4 - \frac{1}{56}\left(\frac{c}{T}\right)^5 + \cdots\right]$$

$-\frac{c}{T+Ht}\left[\frac{1}{6} - \frac{1}{12}(\frac{c}{T+H}) + \frac{1}{20}\left(\frac{c}{T+H}\right)^2 - \frac{1}{30}\left(\frac{c}{T+H}\right)^3 + \frac{1}{42}\left(\frac{c}{T+H}\right)^4 - \frac{1}{56}\left(\frac{c}{T+H}\right)^5 + \cdots\right],$

$H = \frac{2St}{b+t^2},$

we note that

$M2 > 0,$ if $S > 0$ and $M2 < 0,$ if $S < 0.$ (A.8)

Therefore, from (A.7) and (A.8), we obtain

$\hat{h}_{EBL1} - \hat{h}_{EBL2} = \hat{h}_{EBL2} - \hat{h}_{EBL3} > 0,$

that is

$\hat{h}_{EBL1} > \hat{h}_{BL21} > \hat{h}_{EBL3}.$

iv) According to (A.7) and (A.8), we get

$$\lim_{T\to\infty}\left(\hat{h}_{EBL1} - \hat{h}_{EBL2}\right) = \lim_{T\to\infty}\left(\hat{h}_{EBL2} - \hat{h}_{EBL3}\right) = \frac{2m+1}{2S}\lim_{T\to\infty}M2 = 0.$$

Then,

$$\lim_{T\to\infty}\hat{h}_{EBL1} = \lim_{T\to\infty}\hat{h}_{EBL2} = \lim_{T\to\infty}\hat{h}_{EBL3}.$$

v), vi) From (26), we have

$$\hat{h}_{EBE1} - \hat{h}_{EBE2} = \hat{h}_{EBE2} - \hat{h}_{EBE3} = \frac{t(2m-1)}{c^2(b+t^2)}M,$$

therefore, as (i) and (ii), we can prove the relations.

References

[1]. AbdEllah, A.H. (2006). Comparison of estimates using record statistics from Lomax model: Bayesian and non-Bayesian approaches. J. Statistical Research Iran, 3, 139-158.

[2]. AbdEllah, A.H. (2011) Bayesian One Sample Prediction Bounds for the Lomax Distribution. Indian Journal of Pure and Applied Mathematics, 42, 335-356.

[3]. Ahsanullah, M. (1993). On the Record Values from Univariate Distributions. National Institute of Standards and Technology Journal of Research, Special Publications, 12, 1-6.

[4]. Ahsanullah, M. (1995). Introduction to Record Statistics. Nova Science Publishers, Inc. Commack, New York.

[5]. Arnold, C. B., Balakrishnan, N. and Nagaraja, H. N. (1998). Records.Wiley, New York.

[6]. Azimi, R., Yaghmaei, F. and Fasihi, B. (2013). E-Bayesian estimation based on generalized half logistic progressive type-II censored data. International Journal of Advanced Mathematical Sciences, 1(2), 56-63.

[7]. Balasooriya, U. and Balakrishnan, N. (2000). Reliability Sampling Plans for Log-Normal distribution based on progressively censored samples. IEEE Transactio Reliability, 49(2), 199-203.

[8]. Bekker, A., Roux, J. andMostert, P. (2000). A Generalization of the Compound Rayleigh Distribution: Using a Bayesian Methods on Cancer Survival Times. Communications in Statistics—Theory and Methods, 29(7), 1419-1433.

[9]. Calabria, R. and Pulicini, G. (1996). Point estimation under asymmetric loss functions for left-truncated exponential samples. Communications in Statistics—Theory and Methods, 25(3), 585-600.
[10]. Day, D.K. Ghosh, M. and Srinivasan, C. (1987). Simultaneous Estimation of Parameters under Entropy Loss. Journal of Statistical Planning and Inference, 52, 347-363.
[11]. Day, D.K. and P. L. Liu, P.L. (1992). On Comparison of Estimation in Generalized Life Model. Microelectronics Reliability, 32(1-2), 207-221.
[12]. EL-Sagheer, R.M. (2014). Inferences for the generalized logistic distribution based on record statistics. Intelligent Information Management, 6, 171-182.
[13]. EL-Sagheer, R.M and Ahsanullah, M. (2015). Bayesian Estimation Based on Progressively Type-II Censored Samples from Compound Rayleigh Distribution. Journal of Statistical Theory and Applications, 14(2), 107-122.
[14]. Ghitany, M.E. (2001). A compound Rayleigh survival model and its application to randomly censored data. Statistical Papers, 42, 437-450.
[15]. Gupta, L.I. and Gupta, R. (2017). Bayesian and E –Bayesian estimation of the unknown shape parameter of exponentiated inverted Weibull distribution using different loss functions. International Educational Scientific Research Journal, 3(5), 408-416.
[16]. Han, M. (1997). The structure of hierarchical prior distribution and its applications. Chinese Operation Research and Management Science, 6(3), 31-40.
[17]. Han, M. (2011). E-Bayesian estimation of the reliability derived from Binomial distribution. Applied Mathematical Modelling, 35(5), 2419-2424.
[18]. Han, M. (2017). The E-Bayesian and hierarchical Bayesian estimations for the system reliability parameter. Communications in Statistics—Theory and Methods, 46(4), 1606-1620.
[19]. Kızılaslan, F. (2017). The E-Bayesian and hierarchical Bayesian estimations for the proportional reversed hazard rate model based on record values. Journal of Statistical Computation and Simulation, 87(11), 2253-2273.
[20]. Li, D., Wang, J. and Chen, D. (2012). E-Bayesian estimation and hierarchical Bayesian estimation of estate probability in engineering. Systems Engineering Procedia, 5, 349-354.
[21]. Li, X.Y, Shi, Y., Wei, J. and Chai, J. (2007). Empirical Bayes Estimators of Reliability Performances Using LINEX loss under Progressively Type-II Censored Samples. Mathematics and Computers in Simulation, 73(5), 320-326.
[22]. Nagaraja, H. N. (1988). Record values and related statistics. A review. Communication in Statistics-Theory and Methods, 7, 2223-2238.
[23]. Nasiri, P. and Esfandyarifar, H. (2016). E-Bayesian Estimation of the Parameter of the Logarithmic Series Distribution. Journal of Modern Applied Statistical Methods
[24]. Okasha, H.M. (2014). E-Bayesian estimation for the Lomax distribution based on type-II censored data. Journal of the Egyptian Mathematical Society, 22, 489-495.
[25]. Okasha, H.M. and Wang, J. (2016). E-Bayesian estimation for the geometric model based on record statistics. Applied Mathematical Modelling, 40, 658-670.
[26]. Prakash, G. and Singh, D.C. (2008). Shrinkage estimation in exponential type-II Censored data under LINEX loss. Journal of the Korean Statistical Society, 37(1).53-61.
[27]. Raqab, M. Z. (2002). Inferences for generalized exponential distribution based on record statistics. Journal of Statistical Planning and Inference, 104, 339-350.
[28]. Resnick, S.I. (1987). Extreme Values, Regular Variation, and Point Processes. Springer-Verlag, New York.
[29]. Reyad, H.M. and Ahmed, S.O. (2015). E-Bayesian analysis of the Gumbel type-ii distribution under type-ii censored scheme. International Journal of Advanced Mathematical Sciences, 3(2), 108-120.
[30]. Reyad, H.M. and Ahmed, S.O. (2016). Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on type-ii censoring. International Journal of Advanced Mathematical Sciences, 4(1), 10-17.

[31]. Reyad, H.M., Younis, A.M. and Alkhedir, A. A. (2016). Comparison of estimates using censored samples from Gompertz model: Bayesian, E-Bayesian, hierarchical Bayesian and empirical Bayesian schemes. International Journal of Advanced Statistics and Probability, 4(1), 47-61.
[32]. Soliman, A.A. (2005). Estimation of parameters of life from progressively censored data using Burr-XII model,"IEEE Transactions Reliability, 54(1), 34-42.
[33]. Soliman, A.A. (2008). Estimation for Pareto model using general progressive censored data and asymmetric loss," Communications in Statistics—Theory and Methods, 37(9), 1353-1370.
[34]. Soliman, A.A. and Al-Aboud, F.M. (2008). Bayesian inference using record values from Rayleigh model with application. European Journal of Operational Research, 185, 659-672.
[35]. Varian, H. R. (1975). A Bayesian Approach to Real Estate Assessment. North Holland, Amsterdam, 195-208.
[36]. Wang, J., Li, D. and Chen, D. (2012). E Bayesian estimation and hierarchical Bayesian estimation of the system reliability parameter. Systems Engineering Procedia, 3, 282-289.
[37]. Yousefzadeh, F. (2017). E-Bayesian and Hierarchical Bayesian Estimations for the system reliability Parameter Based on Asymmetric Loss Function. Communication in Statistics-Theory and Methods 46(1), 1-8.