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Research Article

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Interpretation of Conic Sections with Side Conditional Extremes

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Abstract Conic sections are formed when you intersect a plane with a right circular cone. In this tutorial, we will learn more about what makes conic sections special. Here we will have a look at three different conic sections ellipses, parabolas, and hyperbolas. All of these geometric figures may be obtained by the intersection a double cone with a plane. All conic sections have equations of the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, where $A, B, C, D, E, F \in \mathbb{R}$. Here we have drawn a different approach to the conic sections that can be drawn by geometric or matrix methods [2]. The drawing of the conic sections, which are rotated at the origin, in the literature. In the paper we do calculation for that conics will be drawn if it is rotated at a displaced point [1].

MSC2000: 49K30, 49K35, 51A05, 51N25.

Keywords Conic Sections, Lagrange Condition.

1. Introduction

Let us start with the equation of the degenerated conic section at the origin.

$$Ax^2+Bxy+Cy^2=t\,;\ \text{where } A, B, C, t \in \mathbb{R}.$$

$$x \to x_1+a,\ y \to y_1+b, \text{ where } a, b \in \mathbb{R}.$$

So we get

$$A(x_1+a)^2+B(x_1+a)(y_1+b)+C(y_1+b)^2=t$$

$$Ax_1^{\,2}+Bx_1y_1+Cy_1^{\,2}+(2Aa+Bb)x_1+(2Cb+Bb)y_1+Aa^2+Bab+Cb^2-t=0$$

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

Basic axes $y=p_1x$ and $y=p_2x$ in origin with displacement

$$\left.\begin{array}{l} y+b=p_1(x_1+a) \\ y+b=p_2(x_1+a) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} y+b=p_1(x+a) \\ y+b=p_2(x+a) \end{array}\right. .$$

2. Main Result

The standard distance function, on the conical section, of the point $A(x,y)$ to displaced point $N(a,b)$ is

$$f(x,y)=(x-a)^2+(y-b)^2$$

Using conic section equation and lagrange multiplier condition on this standard distance function we have

$$Aax+By+D=2(x-a)\lambda$$
$$2Cy+Bx+E=2(y-a)\lambda$$
$$\Leftrightarrow$$
$$(y-b)\left[Aax+By+D=2(x-a)\lambda\right]$$
$$(x-a)\left[2Cy+Bx+E=2(y-a)\lambda\right]$$

Then the following basic axes of rotation and translation are obtained.

$$(m_1x+n_1y+r_1)(m_2x+n_2y+r_2)=0, \text{ where } m_1, m_2, n_1, n_2, r_1, r_2 \in \mathbb{R}.$$

$$m_1m_2x^2+(m_1n_2+m_2n_1)xy+n_1n_2y^2+(m_1r_2+m_2r_1)x+(n_1r_2+n_2r_1)y+r_1r_2=0$$

Thus, we have

$$m_1m_2=A,\ m_1n_2+m_2n_1=2A-2C,\ n_1n_2=B,$$
$$m_1r_2+m_2r_1=Ba-2Ab-E,$$
$$n_1r_2+n_2r_1=2C-Bb+D,\ r_1r_2=Ea-Db$$

So we have proved, the following theorem which is our main theorem, gives the relationship between the point of departure, the basic axes and the coefficients of the conic section.

Theorem 1. Let $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, where $A, B, C, D, E, F \in \mathbb{R}$ be a conic section. Then

i. The following equations apply between the coefficients of the degenerative conical section and the coordinates of the turning point.

$$D=2Aa+Bb, E=Ba+2Cb$$
$$F=Aa^2+Bab+Ca^2-t, \text{ where } a,b,t \text{ are as above.}$$

ii. If $Ax^2+Bxy+Cy^2+Dx+Ey+F=(k_1x+l_1y+s_1)(k_2x+l_2y+s_2)=0$, where k_i, l_i, s_i are real numbers for $i=1,2$

iii. The coefficients of these lines and the conic section are related as

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=(k_1x+l_1y+s_1)(k_2x+l_2y+s_2)$$

iv. If $A=k_1k_2, C=l_1l_2$ and $F=s_1s_2$ then $k_1l_2+k_2l_1=B, k_1s_2+k_2s_1=D$ and $l_1s_2+l_2s_1=E$.

Note: In the above the point $N(a,b)$ is changed with the point $M(-a,-b)$ because if carry to the orijin $x\to x_1+a, y\to y_1+b$ gives the point $M(-a,-b)$.

Now we give a few examples.

Example. Draw the conic section given by the equation

$$5x^2+4xy+2y^2-14x-8y-34=0.$$

Solution.

$$\left.\begin{array}{l}-14=10a+4b\\-8=4a+4b\end{array}\right\}\Leftrightarrow\left.\begin{array}{c}6a=-6\Leftrightarrow a=-1\\-4+4b=-8\Leftrightarrow b=-1\end{array}\right\}\Leftrightarrow M(1,1),$$

$$f(x,y)=(x-1)^2+(y-1)^2$$

$$\left.\begin{array}{r}10x+4y-14=2(x-1)\lambda \\ 4x+4y-8=2(y-1)\lambda\end{array}\right\} \Leftrightarrow \begin{array}{c}(y-1)\left[10x+4y-14=2(x-1)\lambda\right] \\ (x-1)\left[4x+4y-8=2(y-1)\lambda\right]\end{array}$$

$$-4x^2+6xy+4y^2+2x-14y+6=0$$

$$-2(x-2y+1)(2x+y-3)=0 \Leftrightarrow \begin{cases} y=3-2x \\ y=\dfrac{x}{2}+\dfrac{1}{2}\end{cases}$$

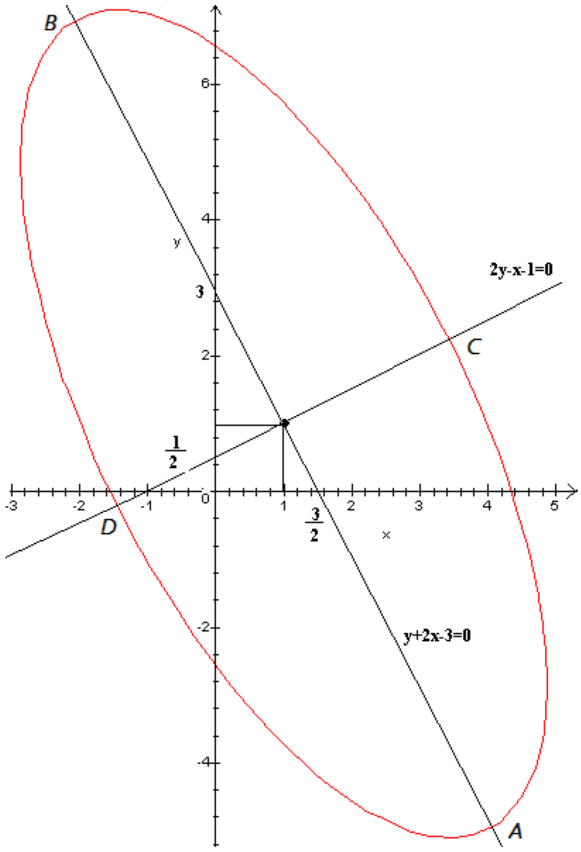
$$y=3-2x \Leftrightarrow 5x^2+4x(3-2x)+24x(3-2x)^2-14x-84x(3-2x)-34=0$$

$$\Leftrightarrow \begin{cases} x_1=4 \Leftrightarrow A(4,-5) \\ x_2=-2 \Leftrightarrow B(-2,7)\end{cases}$$

$$y=\frac{x}{2}+\frac{1}{2} \Leftrightarrow 5x^2+4x\left(\frac{x}{2}+\frac{1}{2}\right)+2\left(\frac{x}{2}+\frac{1}{2}\right)^2-14x-8\left(\frac{x}{2}+\frac{1}{2}\right)-34=0$$

$$\frac{15}{2}x^2-15x-\frac{75}{2}=0 \Leftrightarrow \begin{cases} x_1=1+\sqrt{6} \Leftrightarrow C\left(1+\sqrt{6},\dfrac{1+\sqrt{6}}{2}\right) \\ x_2=1-\sqrt{6} \Leftrightarrow D\left(1-\sqrt{6},\dfrac{1-\sqrt{6}}{2}\right)\end{cases}$$

So we have four critical points as in the figure below.



Example. Draw the conic section given by the equation

$$x^2 - xy + y^2 + 6x - 6y - 48 = 0.$$

Solution.

$$\left.\begin{array}{r} 6 = 2a - 4b \\ -6 = -4a + 2b \end{array}\right\} \Leftrightarrow \left.\begin{array}{r} a = 1 \\ b = -1 \end{array}\right\} \Leftrightarrow M(-1,1)$$

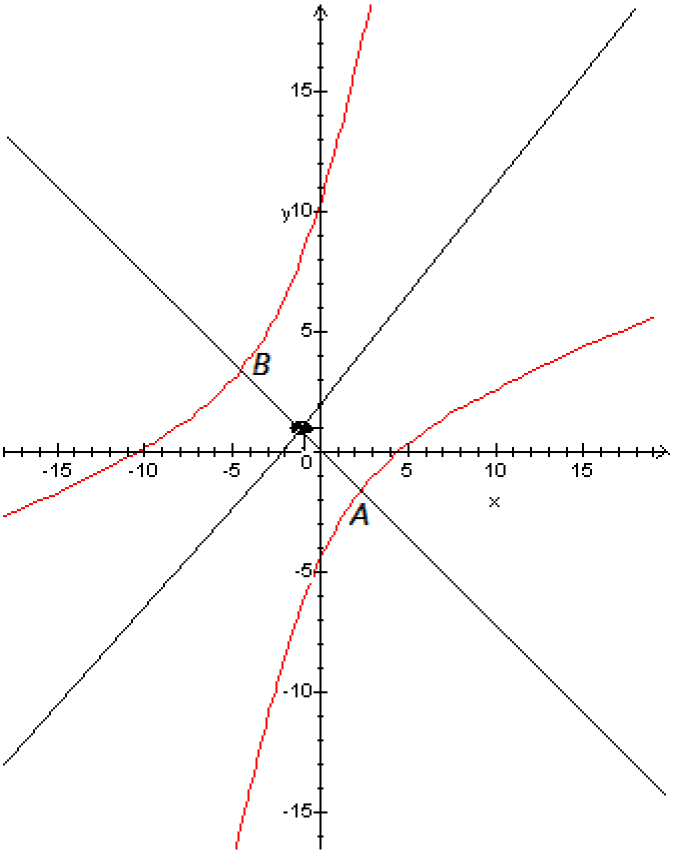
$$f(x,y) = (x+1)^2 + (y-1)^2$$

$$\left.\begin{array}{r} 2x - y + 6 = 2(x+1)\lambda \\ -4x + 2y - 6 = 2(y-1)\lambda \end{array}\right\} \Leftrightarrow \begin{array}{r} (y-1)[2x - y + 6 = 2(x+1)\lambda] \\ (x+1)[-4x + 2y - 6 = 2(y-1)\lambda] \end{array}$$

$$4x^2 - 4y^2 + 8x + 8y = 4(x - y + 2)(x + y) = 0 \Leftrightarrow \begin{cases} y = x + 2 \\ y = -x \end{cases}$$

$$y = -x \Leftrightarrow A(2,-2), B(-4,4)$$

The above two critical points are calculated for hyperbola as below.



Example. Draw the conic section given by the equation

$$x^2 + 2xy + y^2 - \sqrt{2}x + \sqrt{2}y + 2 = 0.$$

Solution.

$$\left.\begin{array}{r} 0 = 2a + 2b \\ 0 = 2a + 2b \end{array}\right\} \Leftrightarrow \left.\begin{array}{r} a = 0 \\ b = 0 \end{array}\right\} \Leftrightarrow y = -x \} \Leftrightarrow M(0,0)$$

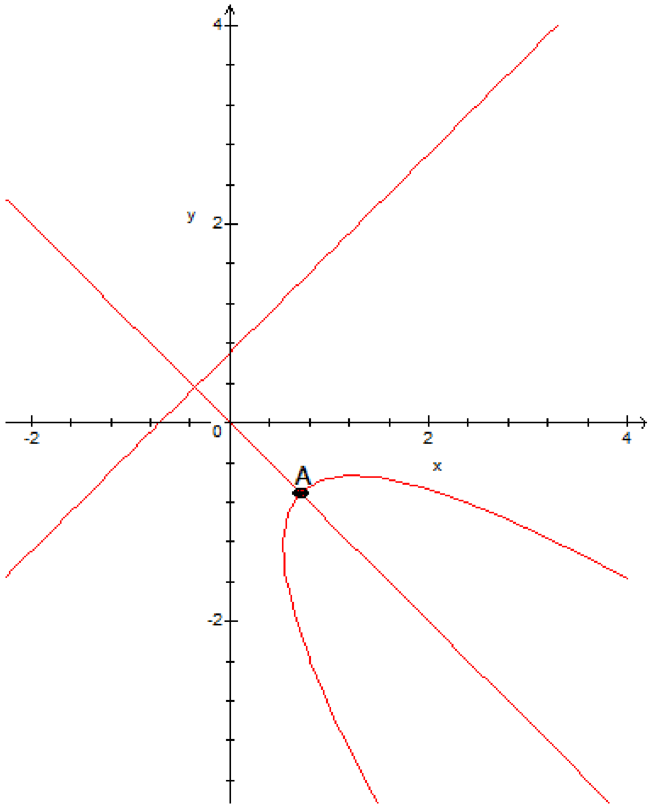
$$f(x,y) = x^2 + y^2$$

$$\left.\begin{array}{l} y\left[2x+2y-\sqrt{2}=2x\lambda\right] \\ x\left[2y+2x+\sqrt{2}=2y\lambda\right] \end{array}\right\} \Leftrightarrow -2x^2+2y^2-\sqrt{2}x-\sqrt{2}y=0$$

$$-2x^2+2y^2-\sqrt{2}x-\sqrt{2}y=-\left(2x-2y+\sqrt{2}\right)\left(y+x\right)=0 \Leftrightarrow \begin{cases} y=-x \\ y=x+\dfrac{\sqrt{2}}{2} \end{cases}$$

$$y=-x \Leftrightarrow x^2+2x(-x)+(-x)^2-\sqrt{2}x+\sqrt{2}(-x)+2=0$$

$$\Leftrightarrow x=\frac{\sqrt{2}}{2} \Leftrightarrow A\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$$



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