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***Journal of Scientific and Engineering Research*, 2017, 4(6):157-160**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

On some 2-color off-diagonal Rado numbers

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Abstract Let ε_0, ε_1 be two equations, each with at least three variables and coefficients not all the same sign. Define the 2-color off-diagonal Rado number $R_2(\varepsilon_0, \varepsilon_1)$ to be the smallest integer N such that for any 2-coloring of $[1, N]$, it must admit a monochromatic solution to ε_0 of the first color or a monochromatic solution to ε_1 of the second color. Motivated by Myers' open problem, we determine the exact numbers $R_2(2x+qy = z, 2x+ y = z)$ and $R_2(2x + 2qy = z, 2x + 2y = z)$ in this paper.

Keywords Schur number, Rado number, off-diagonal Rado number

1. Introduction and Main Results

Let $[a,b]$ denote the set $\left\{x \in \square \,\middle|\, a \le x \le b\right\}$. A function Δ: $[1, n] \rightarrow [0, k-1]$ is called a k-coloring of the set $[1, n]$. Assume that ε is a system of equations in m variables. We say that a solution $x_1, x_2, \ldots, x_m$ to ε is monochromatic if and only if

$$\Delta(x_1) = \Delta(x_2) = \ldots = \Delta(x_m). \tag{1.1}$$

In 1916, Schur [15] proved that for every integer $k \geq 2$, there exists a least integer $n = S(k)$ such that for every k-coloring of the set $[1, n]$, there exists a monochromatic solution to $x+y=z$. The integer $S(k)$ is called Schur number. Rado [10, 11] generalized the work of Schur to arbitrary system of linear equations. For a given equation ε, the least integer n is called k-color Rado number if it exists and for every coloring of the set$[1,n]$ with k colors, there exists a monochromatic solution to ε. If such an integer n does not exist, we say that the k-color Rado number for the equation ε is infinite. In recent years there has been considerable interest in finding the exact Rado numbers for particular linear equations and in several other closely related problems, see for example [1,2,3,4,5,6,9,12,13,14,16].

Let ε_0 and ε_1 be two equations. Define the 2-color off-diagonal Rado number $R_2(\varepsilon_0,\varepsilon_1)$ to be the least integer N (if it exists) for which any 2-coloring of$[1,N]$ must admit a monochromatic solution of color $\boldsymbol{I}$ to ε_i for some $\boldsymbol{i} \in \{0,1\}$. Note that if $\varepsilon_0 = \varepsilon_1$, then the2-color off-diagonal Rado number is nothing but the 2-color Rado number.

Myers and Robertson [8] determined the exact 2-color off-diagonal Rado numbers of the form $R_2(x + qy = z, x + sy = z)$. In the same paper, they also established the lower bound of $R_2(tx + sy = z, tx + qy = z)$, which can be stated as follows.

Theorem 1.1 *Let $q \geq s \geq t$ be positive integers. Then,*

$$R_2(tx + sy = z, tx + qy = z) \geq t(t + q)(t + s) + \frac{\gcd(t,q)}{\gcd(t,s,q)} s. \tag{1.2}$$

In his thesis [7], Myers provided an open problem: what are the precise off-diagonal Rado numbers of the form $R_2(tx + sy = z, tx + qy = z)$? Motivated by this open problem, we shall establish the exact formulas for $R_2(2x + y = z, 2x + qy = z)$ and $R_2(2x + 2y = z, 2x + 2qy = z)$. Throughout this paper, we always let blue and red be the two colors and denoted by 0 and 1, respectively. The main results can be stated as the following two theorems which are proved in the next two sections.

Theorem 1.2 *Let $q \geq 2$ be an integer. We have*

$$R_2(2x + y = z, 2x + qy = z) = \begin{cases} 20, & if \quad q = 2, \\ 3q+8, & if \quad q = 3. \end{cases} \tag{1.3}$$

Theorem 1.3 *If $q \geq 2$ is an integer, then*

$$R_2(2x+2y=z,2x+2qy=z)=16q+18. \tag{1.4}$$

2. Proof of Theorem 1.2.

It is easy to check that Theorem 1.2 holds for q = 2, 3, 4. Therefore, it suffices to consider $q \geq 5$. We first show that

$$R_2(2x + qy = z, 2x + y = z) \geq 3q + 8. \tag{2.1}$$

The lower bound can be established by exhibiting a coloring that avoids red solution to $2x+qy=z$ and blue solution to $2x+y=z$. Consider the 2-coloring of $[1,3q+7]$ defined by coloring $[3, 3q+5]$ red and its complement blue. It is easy to check that the coloring avoids red solution to $2x + qy = z$ and blue solution to $2x + y = z$.

We shall now establish the upper bound, that is,

$$R_2(2x + qy = z, 2x + y = z) \leq 3q + 8. \tag{2.2}$$

Let Δ be a 2-coloring of $[1,3q+8]$ using the colors red and blue. Without loss of generality, we assume, for contradiction, that there is no red solution to $2x + qy = z$ and no blue solution to $2x+y=z$. We break our proof into two cases.

Case 1: $\Delta(1)=0$. It follows from $\Delta(1)=0$ that $\Delta(3)=1$ which yields $\Delta(3q+6)=0$. It follows from $\Delta(1)=0$ and $\Delta(3q+6)=0$ that $\Delta(3q+8)=1$. The facts $\Delta(3)=1$ and $\Delta(3q + 8) = 1$ imply that $\Delta(4) = 0$. Since (1, 2, 4) solves $2x + y = z$, we see that $\Delta(2) = 1$. It follows from $\Delta(3) = 1$ and $\Delta(2) = 1$ that $\Delta(3q + 4) = 0$. Now, we have $\Delta(1)=\Delta(3q+4)=\Delta(3q+6)=0$ and $(1,3q+4,3q+6)$ is a blue solution to $2x+y=z$. This is a contradiction.

Case 2: $\Delta(1) = 1$. $\Delta(1) = 1$ implies that $\Delta(q + 2) = 0$ which yields $\Delta(3q + 6) = 1$. It follows from $\Delta(3q + 6) = 1$ that $\Delta(3) = 0$. Combining $\Delta(3) = 0$ and $\Delta(q + 2) = 0$, we have $\Delta(q + 8) = 1$. The facts $\Delta(q + 8) = 1$ and $\Delta(1) = 1$ imply that $\Delta(4) = 0$. It follows from $\Delta(4) = 0$ and $\Delta(q + 2) = 0$ that $\Delta(q + 10) = 1$. Since $(5, 1, q + 10)$ solves $2x + qy = z$, we see that $\Delta(5) = 0$. Combining $\Delta(q + 2) = 0$ and $\Delta(5) = 0$, we have that $\Delta(q + 12) = 1$. The facts $\Delta(q + 12) = 1$ and $\Delta(1) = 1$ imply that $\Delta(6) = 0$. It follows from $\Delta(6) = 0$ and $\Delta(q + 2) = 0$ that $\Delta(q + 14) = 1$. Since $(7, 1, q + 14)$ is a solution to $2x + qy = z$, we see that $\Delta(7) = 0$ which implies that $\Delta(q + 16) = 1$ or else $(7,q+2,q+16)$ is a blue solution to $2x+y=z$. Now, $\Delta(1)=1$ and $\Delta(q+16)=1$, we see that $\Delta(8) = 0$. It follows from $\Delta(q + 2) = 0$ and $\Delta(8) = 0$ that $\Delta(q + 18) = 1$. Note that $q \geq 5$ implies that $3q + 8 \geq q + 18$. Since $(9, 1, q + 18)$ solves $2x + qy = z$, we see that $\Delta(9) = 0$. Now, we have $\Delta(3) = \Delta(9) = 0$ and (3, 3, 9) is a blue solution to $2x + y = z$, which is a contradiction.

3. Proof of Theorem 1.3.

Employing Theorem 1.1, we obtain the lower bound

$$R_2(2x + 2y = z, 2x + 2qy = z) \geq 16q + 18. \tag{3.1}$$

Now, we turn to establish the upper bound, that is,

$$R_2(2x + 2y = z, 2x + 2qy = z) \leq 16q + 18. \tag{3.2}$$

Let Δ be a 2-coloring of $[1, 16q + 18]$ using the colors red and blue. Without loss of generality, we assume, for contradiction, that there is no red solution to $2x+2qy=z$ and no blue solution to $2x+2y=z$. We also break our proof into two cases.

Case 1: $\Delta(1) = 0$. $\Delta(1) = 0$ implies that $\Delta(4) = 1$ which yields $\Delta(8q + 8) = 0$. The facts $\Delta(1)=0$ and $\Delta(8q+8)=0$ imply that $\Delta(16q+18)=1$. Combining $\Delta(16q+18)=1$ and $\Delta(4) = 1$, we have $\Delta(4q + 9) = 0$. The facts $\Delta(4q + 9) = 0$ and $\Delta(1) = 0$ imply that $\Delta(8q+20)=1$. It follows from $\Delta(8q+20)=1$ and $\Delta(4)=1$ that $\Delta(10)=0$. Combining $\Delta(8q + 8) = 0$ and $\Delta(1) = 0$, we obtain $\Delta(4q + 3) = 1$. The facts $\Delta(4q + 3) = 1$ and $\Delta(4) = 1$ imply that $\Delta(16q + 6) = 0$. It follows from $\Delta(16q + 6) = 0$ and $\Delta(1) = 0$ that $\Delta(8q + 2) = 1$.

Subcase 1: $\Delta(2) = 0$. $\Delta(2) = 0$ implies that $\Delta(8) = 1$ which yields $\Delta(16q + 16) = 0$. Combining $\Delta(16q + 16) = 0$ and $\Delta(2) = 0$, we see that $\Delta(8q + 6) = 1$. The facts $\Delta(8q+ 6) = 1$ and $\Delta(4) = 1$ imply that $\Delta(3) = 0$. Now, we have $\Delta(2) = \Delta(3) = \Delta(10)$ and $(2,3,10)$ is a blue solution to $2x + 2y = z$. This is a contradiction.

Subcase 2: $\Delta(2)=1$.$\Delta(2)=1$ implies that $\Delta(4q+4)=0$. It follows from $\Delta(4q+4)= 0$ and $\Delta(1) = 0$ that $\Delta(2q + 1) = 1$. Now, we have $\Delta(2) = \Delta(2q + 1) = \Delta(8q + 2) = 1$ and $(2q+1,2,8q+2)$ is a red solution $2x+2qy=z$. This is a contradiction.

Case 2: $\Delta(1) = 1$. $\Delta(1) = 1$ implies that $\Delta(2q + 2) = 0$ which yields $\Delta(8q + 8) = 1$. It follows from $\Delta(8q + 8) = 1$ that $\Delta(4) = 0$ which yields $\Delta(16) = 1$. It follows from $\Delta(4) = 0$ and $\Delta(2q + 2) = 0$ that $\Delta(4q + 12) = 1$.

Subcase 1: $\Delta(2) = 1$. Combining $\Delta(2) = 1$ and $\Delta(4q +12) = 1$, we see that $\Delta(6) = 0$. The facts $\Delta(6) = 0$ and $\Delta(2q + 2) = 0$ imply that $\Delta(4q + 16) = 1$. It follows from $\Delta(4q+16)=1$ and $\Delta(2)=1$ that $\Delta(8)=0$. The fact $\Delta(2)=1$ implies that $\Delta(4q+4)=0$. Combining $\Delta(4q + 4) = 0$ and $\Delta(4) = 0$, we see that $\Delta(8q + 16) = 1$. The facts $\Delta(8q + 16) = 1$ and $\Delta(2) = 1$ imply that $\Delta(2q + 8) = 0$. It follows from $\Delta(2q + 8) = 0$ and $\Delta(8) = 0$ that $\Delta(4q + 32) = 1$. Note that $4q + 32 < 16q + 18$ since $q \geq 2$. Now, we have $\Delta(16)=\Delta(4q+32)=\Delta(2)=1$ and $(16,2,4q+32)$ is a red solution to $2x+2qy=z$. This is a contradiction.

Subcase2:$\Delta(2)=0$. It follows from $\Delta(2)=0$ and $\Delta(2q+2)=0$ that $\Delta(4q+8)=1$. Combining $\Delta(4q + 8) = 1$ and $\Delta(1) = 1$, we get $\Delta(q + 4) = 0$. The facts $\Delta(2) = 0$ and $\Delta(q + 4) = 0$ imply that $\Delta(2q + 12) = 1$. It follows from $\Delta(2q + 12) = 1$ and$\Delta(1) = 1$ that $\Delta(6) = 0$. $\Delta(2) = 0$ implies that $\Delta(8) = 1$. Combining $\Delta(8) = 1$ and $\Delta(1)=1$, we have $\Delta(2q+16)=0$. Since $(q+2,6,2q+16)$ solves $2x+2y=z$, we see that $\Delta(q+2)=1$. This implies that $\Delta(4q+4)=0$ or else $(q+2,1,4q+4)$ is a red solution to $2x + 2qy = z$. It follows from $\Delta(4q + 4) = 0$ and $\Delta(q + 4) = 0$ that $\Delta(10q + 16) = 1$. Now, we have $\Delta(1) = \Delta(4q + 8) = \Delta(10q + 16) = 1$ and $(4q + 8, 1, 10q + 16)$ is a red solution to $2x+2qy=z$. This is a contradiction.

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