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***Journal of Scientific and Engineering Research*, 2017, 4(5):82-85**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

A Note on Weighted Harmonic Brgman Functions on the Upper Half-Space

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Abstract On the setting of the upper half-space **H** of the Euclidian n-space, we give size estimates of partial derivatives of weighted harmonic Bergman functions. Also, we show that for any multi-index $\vec{\beta}$, there is a function $u \in b_\alpha^p$ satisfying $D^{\vec{\beta}} u \notin b_\alpha^p$.

Keywords Weighted Harmonic Bergman Functions, Upper Half-Space, Cauchy's Estimates, Poisson Kernel.

1. Introduction

For a fixed positive integer $n \geq 2$, let $\mathbf{H} = \mathbf{R}^{n-1} \times (0, \infty)$ be the upper half-space. We write point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$, $1 \leq p < \infty$, and $\Omega \subset \mathbf{R}^n$, let $b_\alpha^p(\Omega)$ denote *weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on Ω such that

$$\|u\|_{b_\alpha^p(\Omega)} := \left(\int_\Omega |u(z)|^p \, dV_\alpha(z)\right)^{1/p} < \infty,$$

Where $dV_\alpha(z)$ = $\mathrm{dist}(z, \partial\Omega)^\alpha dz$, $\mathrm{dist}(z, \partial\Omega)$ denotes the Euclidian distance from z to the boundary of Ω and dz is the Lebesgue measure on $\mathbf{R}^n$. We let $b_\alpha^p = b_\alpha^p(\mathbf{H})$. Then we can check that $dV_\alpha(z) = z_n^\alpha dz$ on **H** and space b_α^p is a Banach space.

Harmonic Bergman spaces are not studied as extensively as their holomorphic counterparts and many results on Bergman spaces has been done for bounded domains. [4] and [9], for example, are good references for holomorphic Bergman spaces. $b_0^p(\Omega)$ is studied in [8] and [5] on the setting of upper half-space and bounded smooth domain in $\mathbf{R}^n$, respectively. $b_\alpha^p(B)$ where $B$is the open unit ball in $\mathbf{R}^n$ is studied in [3]. Also, we studied weighted harmonic Bergman spaces in [6] and [7].

This paper is organized as follows. In section 2, we review preliminary results about the extended Poisson kernel and the Cauchy's estimates for harmonic functions. Also, we give size estimate of L^p-norm of partial derivatives of extended Poisson kernel. In section 3 is devoted to prove several properties of functions in b_α^p and its derivatives.

Constants. Throughout the paper we use the same letter C to denote various constants which may change at each occurrence. The constant C may often depend on the dimension n and some other parameters, but it is always independent of particular functions, point or parameters under consideration. For nonnegative quantities A and B, we often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some inessential positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $B \gtrsim A$.

2. Preliminary Results

In this section, we first review some preliminary results about the extended Poisson kernel on the upper half-space.

Let $B(z, r)$ be the open ball in $\mathbf{R}^n$ centered at $z \in \mathbf{R}^n$ with radius $r > 0$ and let $P(z, \omega)$ be the extended Poisson kernel on **H**, i.e.,

$$P_z(\omega) := P(z,\omega) = \frac{2}{nV(B(0,1))}\frac{z_n+\omega_n}{|z-\bar{\omega}|^n}, \qquad (2.1)$$

where $z \in \mathbf{H}$, $\omega \in \overline{\mathbf{H}}$, $\bar{\omega} = (\omega', -\omega_n)$ and $V(B(0,1))$ is the volume of the unit ball in $\mathbf{R}^n$. Then it is well known that for each $z \in$ H and for every $\omega \in \overline{\mathbf{H}}$.

$$\int_{\partial\mathbf{H}} P(z,\omega)d\omega' = 1. \qquad (2.2)$$

Here $\partial\mathbf{H} = \mathbf{R}^{n-1}$ denote the boundary of **H**. See [1] for details.

From (2.1), we see that for multi-indices $\vec{\beta} = (\beta_1, \dots, \beta_n)$ and $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$,

$$D_z^{\vec{\beta}} D_\omega^{\vec{\gamma}} P(z,\omega) = D_{z_1}^{\beta_1} \cdots D_{z_n}^{\beta_n} D_{\omega_1}^{\gamma_1} \cdots D_{\omega_n}^{\gamma_n} P(z,\omega)$$
$$= (-1)^{\gamma_1+\cdots+\gamma_{n-1}} D_{z_1}^{\beta_1+\gamma_1} \cdots D_{z_n}^{\beta_n+\gamma_n} P(z,\omega)$$
$$= (-1)^{\gamma_1+\cdots+\gamma_{n-1}} \frac{f(z-\bar{\omega})}{|z-\bar{\omega}|^{n+2|\vec{\beta}|+2|\vec{\gamma}|}}, \qquad (2.3)$$

where $f = f_{\vec{\beta},\vec{\gamma}}$ is a homogeneous polynomial of degree $1+|\vec{\beta}|+|\vec{\gamma}|$. (Here β_j, γ_j are nonnegative integers for each j=1,…,n and $|\vec{\beta}| = \beta_1 + \cdots + \beta_n$.)

In the following lemma, we give estimate L^p-norm size of $D_z^{\vec{\beta}} P(z,(0,\delta))$ for $\delta > 0$.

Lemma 2.1. Let$\delta > 0$, $\vec{\beta}$ be a multi-index satisfying $(n+|\vec{\beta}|-1)p > n+\alpha$ and let

$$u(z) = D_z^{\vec{\beta}} P(z,(0,\delta))$$

on ***H***. Then we have

$$\|u\|_{b_\alpha^p} \approx \delta^{(n+\alpha)/p-n-|\vec{\beta}|+1}.$$

Proof. First note from (2.3) that

$$u(z) = \frac{f(z+(0,\delta))}{|z+(0,\delta)|^{n+2|\vec{\beta}|}}$$

for some homogenous polynomial f of degree $1+|\vec{\beta}|$. Then we see from the change of variables $z \mapsto \delta z$ and the homogeneity of f that

$$\|u\|_{b_\alpha^p}^p = \int_{\mathbf{H}} \frac{|f(z+(0,\delta))|^p}{|z+(0,\delta)|^{p(n+2|\vec{\beta}|)}} z_n^\alpha dz$$
$$= \frac{\delta^{p(|\vec{\beta}|+1)+n+\alpha}}{\delta^{p(n+2|\vec{\beta}|)}} \int_H \frac{|f(z+(0,1))|^p}{|z+(0,1)|^{p(n+2|\vec{\beta}|)}} z_n^\alpha dz. \qquad (2.4)$$

Let I denote the integral in (2.4). Because f is a polynomial of degree $1+|\vec{\beta}|$, we have from (2.2) that

$$0 < I \lesssim \int_H \frac{z_n^\alpha}{|z+(0,1)|^{(n+|\vec{\beta}|-1)p}} dz$$
$$\lesssim \int_0^\infty \frac{z_n^\alpha}{(z_n+1)^{(n+|\vec{\beta}|-1)p-n+1}} \int_{\partial H} \frac{z_n+1}{|z+(0,1)|^n} dz' dz_n$$
$$\lesssim \int_0^\infty \frac{1}{(z_n+1)^{(n+|\vec{\beta}|-1)p-n-\alpha+1}} \int_{\partial H} P(z,(0,1))\, dz' dz_n < \infty,$$

where we used the fact $(n+|\vec{\beta}|-1)p > n+\alpha$.

Therefore, we have

$$\||u|\|_{b_\alpha^p} \approx \delta^{(n+\alpha)/p-n-|\vec{\beta}|-1},$$

as desired and this completes the proof.

Lemma 2.2 which is called Cauchy's estimates for harmonic functions is proved in [1]

Lemma 2.2 *If $\vec{\beta}$ be a multi-index, then*

$$|D^{\vec{\beta}} u(z)| \lesssim \frac{M}{r^{|\vec{\beta}|}}$$

for all functions u harmonic and bounded by M on $B(z,r)$

3. Main Result

Before we prove the main results, we first show one lemma which implies point evaluation is a bounded linear functional on b_α^p and convergence in b_α^p-norm implies the uniform convergence on each compact subset of **H**.

Lemma 3.1 *Let $u \in b_\alpha^p$ and let $z \in$* **H.** *Then we have*

$$|u(z)| \lesssim \frac{\|u\|_{b_\alpha^p}}{z_n^{(n+\alpha)/p}}.$$

Proof. Let $r = z_n/2$ and apply the volume version of the mean-value property to u on $B(z,r)$. After taking Jensen's inequality, we have

$$\begin{aligned} |u(z)|^p &= \left|\frac{1}{V(B(z,r))}\int_{B(z,r)} u(w)dw\right|^p \\ &\lesssim \frac{1}{V(B(z,r))}\int_{B(z,r)} |u(w)|^p dw \\ &\approx \frac{1}{z_{\mathrm{n}}^{\mathrm{n}}}\int_{B(z,r)} |u(w)|^p \frac{w_n^\alpha}{z_n^\alpha} dw. \end{aligned}$$

Therefore we have

$$|u(z)| \lesssim \frac{\|u\|_{b_\alpha^p}}{z_n^{(n+\alpha)/p}},$$

as desired.

From Lemma 2.2 and Lemma 3.1, we estimate size of partial derivatives of functions in b_α^p.

Theorem 3.2. Let $u \in b_\alpha^p$ and let $\vec{\beta}$ be a multi-index. Then we have

$$\left|D^{\vec{\beta}} u(z)\right| \leq \frac{\|u\|_{b_\alpha^p}}{z_n^{|\vec{\beta}|+(n+\alpha)/p}}$$

For $z \in$ **H**.

Proof. Fix $z \in$ **H** and let $\mathrm{r} = z_n/2$. Then we know from Lemma 3.1 that for $\mathrm{w} \in B(z,r)$,

$$|u(w)| \lesssim \frac{\|u\|_{b_\alpha^p}}{w_n^{(n+\alpha)/p}} \approx \frac{\|u\|_{b_\alpha^p}}{z_n^{(n+\alpha)/p}},$$

because $w_n \approx z_n$ on $B(z,r)$. Therefore we get from Lemma 2.2 that

$$\left|D^{\vec{\beta}} u(z)\right| \lesssim \frac{1}{r^{|\vec{\beta}|}} \cdot \frac{\|u\|_{b_\alpha^p}}{z_n^{(n+\alpha)/p}} \approx \frac{\|u\|_{b_\alpha^p}}{z_n^{|\vec{\beta}|+(n+\alpha)/p}}.$$

This completes the proof

In the following theorem we show that for any multi-index $\vec{\beta} \neq \vec{0}$ and for any $p \in [1,\infty)$, there exists a function $u \in b_\alpha^p$ satisfying that $D^{\vec{\beta}} u$ does not belong to b_α^p.

Theorem 3.3. Let $\vec{\beta}$ be a nonzero multi-index and let $1 \leq p < \infty$. Then there is a function $u \in b_\alpha^p$ such that $D^{\vec{\beta}} u \notin b_\alpha^p$.

Proof. To derive a contradiction, suppose that for all $u \in b_\alpha^p$, $D^{\vec{\beta}} u \in b_\alpha^p$. Then we see from Theorem 3.2 and the closed graph theorem that the map $u \mapsto D^{\vec{\beta}} u$ is a bounded map on b_α^p.

First, we choose a nonnegative integer k satisfying

$$(n + k - 1)\mathrm{p} > n + \alpha$$

and let $u(z) = D_n^k P(z,0)$ for $z \in$ **H.** Then we see from (2.3) that u is harmonic on **H** and

$$u(z) = \frac{f(z)}{|z|^{n+2k}}, \quad D^{\vec{\beta}}u(z) = \frac{g(z)}{|z|^{n+2k+2|\vec{\beta}|}},$$

where f and g are homogenous polynomial of degree $1+k$ and $1 + k + |\vec{\beta}|$, respectively.
Let $u_\delta(z) = u(z + (0,\delta))$ for $\delta > 0$. Then we see Lemma 2.1 that

$$\|u_\delta\|_{b^p_\alpha} \approx \delta^{(n+\alpha)/p-n-k+1}.$$

Note that $|\vec{\beta}| > 0$ and so $(n + k + |\vec{\beta}| - 1)p > n + \alpha$. Then we also see from Lemma 2.1 that

$$\left\|D^{\vec{\beta}}u_\delta\right\|_{b^p_\alpha} \approx \delta^{(n+\alpha)/p-n-k-|\vec{\beta}|+1}.$$

These yield that

$$\frac{\left\|D^{\vec{\beta}}u_\delta\right\|_{b^p_\alpha}}{\|u_\delta\|_{b^p_\alpha}} \approx \delta^{-|\vec{\beta}|} \to \infty$$

as $\delta \to 0$. Hence the map $u \longmapsto D^{\vec{\beta}}u$ is not bounded on b^p_α, which is a contradiction. Therefore the proof is complete.

Acknowledgments
The work reported in this paper was conducted during sabbatical year of Kwangwoon University in 2016.

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