Available online www.jsaer.com

***Journal of Scientific and Engineering Research*, 2017, 4(5):291-307**

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Comparison Inequalities for heat semigroups and heat kernels

Arabi A Aamer Arabi[1], Osman A Adam Osman[2]

[1]Department of Mathematics-College of Science Kordofan University Sudan
[2]Department of Mathematics-College of Science & Arts in Oklat Alskoor – Qassim University - Saudi Arabia

Abstract We demonstrate the inequality of the solutions of the number of heat equations associated with the regular Dirichlet, form, this disparity we obtain the differences in the inequality of the semi groups and the heat kernels which are used to obtain the point wise estimates of heat kernels.

Keywords Heat kernel, open set, Manifolds, Dirichlet, subset, Hilbert space, subspace, Inequalities

Introduction

In this paper, we are concerned with certain inequalities involving heat kernels on arbitrary metric measure spaces. The motivation comes from the following three results.

(i). Let M be a Riemannian manifold and $p_t(x,y)$ be the heat kernel on M associated with the Laplace-Beltrami operator Δ. Let $\{X_t\}_{t\geq 0}$ be the diffusion process generated by Δ. For any open set Ω, denote by $\psi_\Omega(t,x)$ the probability that X_t exits from Ω before the time t, provided $X_0 = x$. It was proved in [1] that, for any two disjoint open subsets U and V of M and for all $x \in U, y \in V$, $t,s > 0$,

$$p_{t+s}(x,y) \leq \psi_U(t,x) \sup_{\substack{s\leq t'\leq t+s \\ u\in\partial U}} p_{t'}(u,y) + \psi_V(s,y) \sup_{\substack{t\leq t'\leq t+s \\ v\in\partial V}} p_{t'}(v,x)$$

(1)

(see Fig. 1). Similarly, if $U \subset V$ then, for all $x \in U$ and $y \in V$,

$$p_{t+s}(x,y) \leq p^V_{t+s}(x,y) + \psi_U(t,x) \sup_{\substack{s\leq t'\leq t+s \\ u\in\partial U}} p_{t'}(u,y) + \psi_V(s,y) \sup_{\substack{t\leq t'\leq t+s \\ v\in\partial V}} p_{t'}(v,x),$$

(2)

where $p^V_t(x,y)$ is the heat kernel in V with the Dirichlet boundary condition in ∂V (see Fig. 2). The estimates (1) and (2) were used in [1] to obtain heat kernel bounds on manifolds with ends.

(ii). Let now $\{X_t\}_{t\geq 0}$ be a diffusion process on a metric measure space (M,d,μ), and assume that $\{X_t\}$ possesses a continuous transition density $p_t(x,y)$ that will be called the heat kernel. It was proved in [2] that, for any open set $V \subset M$ and for all $x \in U, y \in V, t,s > 0$,

$$p_{2t}(x,x) \leq p^V_{2t}(x,x) + 2\psi_V(t,x) \sup_{v\in V} p_t(v,v).$$

(3)

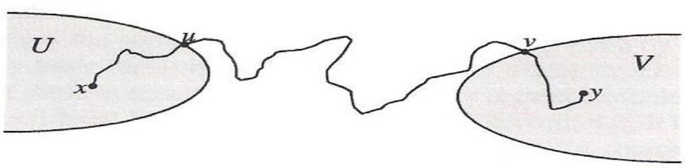

Figure 1: Any sample path, connecting x and y, either exits from the set U before time t when starting at x, or exits from the set V before time s when starting at y.

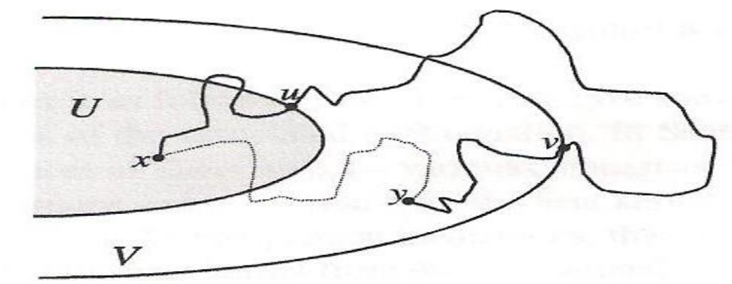

Figure 2: Any sample path, connecting x and y, either stays in V, or exits from the set U before time t when starting at x, or exits from the set V before time s when starting at y.

In the setting of manifolds, one sees that (3) is a particular case of (2) where $U = V$ and $x = y$ since

$$\sup_{\substack{t\leq t'\leq 2t \\ t\in\partial V}} p_{t'}(v,v) \leq \sup_{v\in V} p_t(v,v).$$

Kigami used (3) in [2] to develop a technique for obtaining an upper bound of $p_t(x,x)$, given a certain estimate of the Dirichlet heat kernel $p_t^V(x,x)$.He then applied this technique to obtain heat kernel estimates on post-critically finite self-similar fractals.

iii. In the previous setting, but without the continuity of the heat kernel, the authors proved in [3] the following inequality:

$$\operatorname{esup}_{y\in V} p_{t+s}(x,y) \leq \operatorname{esup}_{y\in V} p_t^V(x,y) + \psi_V(t,x)\operatorname{esup}_{y,z\in V} p_s(y,z) \quad (4)$$

for all $t,s > 0$ and almost all $x \in V$, where esup stands for the essential supremum.

We first recall some terminology from the theory of Dirichlet form [4], and prove some further properties of Dirichlet forms, which are of independent interest for their own right.

Let (M,d,μ) be a *metric measure space*, that is, the couple (M,d) is a locally compact separable metric space and μ is a Radon measure on M with a full support, that is, $\mu(\Omega) > 0$ for any non-empty open subset Ω of M. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in $L^2 := L^2(M,\mu)$, that is, $\mathcal{F}$ is a dense subspace of L^2 and $\mathcal{E}(f,g)$ is a bilinear, symmetric, non-negative definite, closed, and Markovian functional on $\mathcal{F}\times\mathcal{F}$. The closeness of $(\mathcal{E},\mathcal{F})$ means that $\mathcal{F}$ is a Hilbert space with the norm $\left(\|f\|_2^2 + \mathcal{E}(f)\right)^{1/2}$, where $\|\cdot\|_2$ is the norm of $L^2(M,\mu)$ and $\mathcal{E}(f) := \mathcal{E}(f,f)$. The Markovian property means that $f \in \mathcal{F}$ implies $\tilde{f} := (f \vee 0)\wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\tilde{f}) \leq \mathcal{E}(f)$.

Let Δ be the generator of $(\mathcal{E},\mathcal{F})$, that is, an operator in L^2 with the maximal domain dom $(\Delta) \subset \mathcal{F}$ such that

$$\mathcal{E}(f,g) = -(\Delta f, g) \quad \text{for all } f \in \mathrm{dom}(\Delta), g \in \mathcal{F}.$$

Then Δ is a non-positive definite self-adjoint operator in L^2. Let $\{P_t\}_{\{t\geq 0\}}$ be the heat semigroup associated with the form $(\mathcal{E},\mathcal{F})$, that is, $P_t = exp(t\Delta)$. It follows that, for any $t \geq 0, P_t$ is a bounded self-adjoint operator in L^2. The relation between P_t and Δ is given also by the identity

$$\Delta f = L^2 - \lim_{t\to 0}\frac{1}{t}(P_t f - f),$$

where the limit exists if and only if $f \in \mathrm{dom}(\Delta)$. A similar relation takes place between P_t and $\mathcal{E}$:

$$\mathcal{E}(f,g) = \lim_{t\to 0}\frac{1}{t}(f - P_t f, g),$$

for all $f,g \in \mathcal{F}$. The heat semigroup $\{P_t\}$ of a Dirichlet form is always Markovian, that is, for any $0 \leq f \leq 1$ a.e. in M, we have that $0 \leq P_t f \leq 1$ a.e. in M for any $t > 0$.

A family $\{p_t\}_{t>0}$ of $\mu\times\mu$-measurable functions on $M \times M$ is called the heat kernel of the Dirichlet form $(\mathcal{E},\mathcal{F})$ if p_t is the integral kernel of the operator P_t, that is, for any $t > 0$ and for any $f \in L^2(M,\mu)$,

$$P_t f(x) = \int_M p_t(x,y) f(y) d\mu(y) \quad (5)$$

for μ-almost all $x \in M$. The form $(\mathcal{E},\mathcal{F})$ is *regular* if the space $\mathcal{F}\cap C_0(M)$ is dense both in $\mathcal{F}$ and in $C_0(M)$, where $C_0(M)$ is the space of all real-valued continuous functions in M with compact support. For any two subsets U,Ω $(U \Subset \Omega)$ of M, a *cut-off function* ϕ for the pair (U, Ω) is a function in $\mathcal{F}\cap C_0(M)$ such that $0 \leq \phi \leq 1$ in M, $\phi = 1$ in an open neighborhood of $\overline{U}$, and supp $(\phi) \subset \Omega$. If $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form, then a cut-off function exists for any pair (U, Ω) provided that Ω is open and $\overline{U}$ is a non-empty compact subset of Ω [4].

Let Ω be a non-empty open subset of M. We identify the space $L^2(\Omega)$ as a subspace of $L^2(M)$ by extending any function $f \in L^2(\Omega)$ to M by setting $f = 0$ outside Ω. Denote by $\mathcal{F}(\Omega)$ the closure of $\mathcal{F} \cap C_0(\Omega)$ in F-norm. It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega)$ [4]. We refer to $(\mathcal{E}, \mathcal{F}(\Omega))$ as a *restricted* Dirichlet form. Denote by $\{P_t^{\Omega}\}_{t\geq 0}$ the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$. It is known that, for any two open subsets $\Omega_1 \subset \Omega_2$ of M, for any $0 \leq f \in L^2$, and for any $t > 0$,

$$P_t^{\Omega_1} f \leq P_t^{\Omega_2} f \ \text{ a.e. in } M.$$

Also, if $\{\Omega_k\}_{k=1}^{\infty}$ is an increasing sequence of open sets [5]. And $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ then, for any $t > 0$,

$$P_t^{\Omega_k} f \to P_t^{\Omega} f \ \text{ a.e. in } M \text{ as } k \to \infty$$

The form $(\mathcal{E}, \mathcal{F})$ is called local if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with disjoint compact supports in M.

For $0 \leq \rho < \infty$, the form $(\mathcal{E}, \mathcal{F})$ is said to be ρ-*local* if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with compact supports in M and such that

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) > \rho.$$

In particular, if $\rho = 0$ then the ρ-local is the same as the local. We say that the form $(\mathcal{E}, \mathcal{F})$ is quasi-local if it is ρ-local for some $\rho \geq 0$.

Let Ω be an open subset of M and I be an open interval in $\mathbb{R}$. A path $u: I \to L^2(\Omega)$ is said to be weakly differentiable at if, for any $\varphi \in L^2(\Omega)$, the function $(u(\cdot), \varphi)$ is differentiable at t, that is, the limit

$$\lim_{\varepsilon \to 0} \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}, \varphi \right)$$

exists. If this is the case then it follows from the principle of uniform boundedness that there is a (unique) function $w \in L^2(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}, \varphi \right) = (w, \varphi),$$

for all $\varphi \in L^2(\Omega)$. We refer to the function w as the weak derivative of u at t and write $w = \frac{\partial u}{\partial t}$.

A path $u : I \to \mathcal{F}$ is called a weak subsolution of the heat equation in $I \times \Omega$, if the following two conditions are fulfilled :

(i) the path $t \mapsto u(t)|_{\Omega}$ is weakly differentiable in $L^2(\Omega)$ at any $t \in I$;

(ii) for any non-negative $\varphi \in \mathcal{F}(\Omega)$, we have

$$\left(\frac{\partial u}{\partial t}, \varphi \right) + \mathcal{E}(u, \varphi) \leq 0. \tag{6}$$

Similarly one can define the notions of weak super solution and weak solution of the heat equation.

Note that, for any $f \in L^2(\Omega)$, the function $P_t^{\Omega} f$ is a weak solution in $(0, \infty) \times \Omega$ [356], and hence, in $(0, +\infty) \times U$ for any open subset $U \subset \Omega$.

We use the following notation:

$$f_+ := f \vee 0 \quad \text{and} \quad f_- = -(f \wedge 0).$$

Denote by the sign $\overset{\mathcal{H}}{\rightharpoonup}$ a weak convergence in a Hilbert space $\mathcal{H}$ and by $\overset{\mathcal{H}}{\to}$ the strong (norm) convergence in $\mathcal{H}$ .The following statements will be used in this paper.

Proposition(1) [3]. Let $\{u_k\}$ be a sequence of functions in $\mathcal{F}$F such that $u_k \overset{L^2}{\to} u \in \mathcal{F}$ as $k \to \infty$. If in addition the sequence $\{\mathcal{E}(u_k)\}$ is bounded, then $u_k \overset{\mathcal{F}}{\rightharpoonup} u$ as $k \to \infty$.

Proposition(2)[4]. Any Dirichlet form $(\mathcal{E}, \mathcal{F})$ possesses the following properties

(a) If $u, v \in \mathcal{F}$, then all the functions $u \wedge v, u \vee v, u \wedge 1, u_+, u_-, |u|$ also belong to $\mathcal{F}$

(b) If $u, v \in \mathcal{F} \cap L^{\infty}(M)$, then $uv \in \mathcal{F}$.

(c) If $0 \leq u \in \mathcal{F}$, then $u \wedge n \overset{\mathcal{F}}{\to} u$ as $n \to \infty$.

(d) *Let $\phi(s)$ be a Lipschitz function on $\mathbb{R}$ such that $\phi(0) = 0$. Then, for any $u \in \mathcal{F}$, $\phi(u) \in \mathcal{F}$ also. Moreover, if $\{u_n\}_{n=1}^{\infty}$ is a sequence of functions from $\mathcal{F}$ and $u_n \overset{\mathcal{F}}{\to} u \in \mathcal{F}$*

as $n \to \infty$, *then* $\phi(u_n) \xrightarrow{\mathcal{F}} \phi(u)$. *Furthermore, if* $\phi(u) = u$ *then* $\phi(u_n) \xrightarrow{\mathcal{F}} \phi(u)$.

Proposition(3) [5]. *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form, and let* $u \in \mathcal{F}$ *and* Ω *be an open subset of M. Then the following are equivalent:*

(i) $u_+ \in \mathcal{F}(\Omega)$.

(ii) $u \leq v$ *in M for some function* $v \in \mathcal{F}(\Omega)$.

Proposition(4) *(parabolic maximum principle).* [7]. *Assume that* $(\mathcal{E}, \mathcal{F})$ *is a regular Dirichlet form in* L^2 *. For* $T \in (0, +\infty]$ *and for an open subset* Ω *of M, let u be a weak subsolution of the heat equation in (0, T)* $\times\, \Omega$ *satisfying the following boundary and initial conditions:*

(i) $u_+(t,\cdot) \in \mathcal{F}(\Omega)$ *for any* $t \in (0, T)$;

(ii) $u_+(t,\cdot) \xrightarrow{L^2(\Omega)} 0$ *as* $t \to 0$.

Then $u(t,x) \leq 0$ *for any* $t \in (0, T)$ *and μ-almost all* $x \in \Omega$.

Next we prove further some general results on Dirichlet forms that will be used later on and are of independent interest.

Proposition(5). *Let* Ω *be a non-empty open subset of M. Then, for any non-negative* $f \in L^2(\Omega)$, *the path* $u(t) = P_t^{\Omega} f$ *is a weak subsolution of the heat equation in* $(0, \infty) \times M$.

Proof*:* We know that $u(t)$ is weakly differentiable in t in $L^2(\Omega)$. Let us show that $u(t)$ is weakly differentiable also in $L^2(M)$. Indeed, for any function $\phi \in L^2(M)$, we have

$$\left(\frac{u(t+s)-u(t)}{s}, \varphi\right) = \left(\frac{u(t+s)-u(t)}{s}, \varphi \mathbf{1}_{\Omega}\right) + \left(\frac{u(t+s)-u(t)}{s}, \varphi \mathbf{1}_{\Omega^c}\right). \quad (7)$$

Since $\varphi \mathbf{1}_{\Omega} \in L^2(\Omega)$, the first term in the right hand side of (7) converges to $\left(\frac{\partial u}{\partial t}, \varphi \mathbf{1}_{\Omega}\right)$ where $\frac{\partial u}{\partial t}$ is the weak derivative in $L^2(\Omega)$. The second term is obviously 0, whence the convergence of the whole sum to $\left(\frac{\partial u}{\partial t}, \varphi\right)$ follows.

Next, let us show that, for any non-negative $\psi \in \mathcal{F}$,

$$\left(\frac{\partial u}{\partial t}, \psi\right) + \mathcal{E}(u, \psi) \leq 0 \text{ for any } t > 0. \quad (8)$$

Indeed, noting that $P_s u(t) \geq P_s^{\Omega} u(t) = u(t)$, we obtain as $s \to 0+$ that

$$\mathcal{E}_s(u,\psi) = \frac{1}{s}(u - P_s u, \psi) \leq \frac{1}{s}(u - P_s^{\Omega} u, \psi) = \frac{1}{s}(u(t) - u(t+s), \psi) \to \left(-\frac{\partial u}{\partial t}, \psi\right).$$

Since $\mathcal{E}_s(u,\psi) \to \mathcal{E}(u,\psi)$ as $s \to 0$, the desired inequality (8) follows. □

The following proposition will be used to prove Proposition(8).

Proposition(6). *Let* Ω_1, Ω_2 *be two non-empty open subsets of M. Then*

$$\mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2) = \mathcal{F}(\Omega_1 \cap \Omega_2). \quad (9)$$

Proof*:* Since $\mathcal{F}(\Omega_1 \cap \Omega_2) \subset \mathcal{F}(\Omega_i)$ for $i = 1, 2,$ we see that

$$\mathcal{F}(\Omega_1 \cap \Omega_2) \subset \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2).$$

To prove the opposite inclusion, we need to verify that $f \in \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2)$ implies $f \in \mathcal{F}(\Omega_1 \cap \Omega_2)$. Assume first that $f \geq 0$. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two sequences from $\mathcal{F} \cap C_0(\Omega_1)$ and $\mathcal{F} \cap C_0(\Omega_2)$, respectively, that both converge to f in $\mathcal{F}$ -norm. As $f \geq 0$ and, hence, $f_+ = f$, it follows from Proposition(2) that

$$(f_k)_+ \xrightarrow{\mathcal{F}} f \text{ and } (g_k)_+ \xrightarrow{\mathcal{F}} f \text{ as } k \to \infty. \quad (10)$$

Since $(f_k)_+ \in \mathcal{F} \cap C_0(\Omega_1)$ and $(g_k)_+ \in \mathcal{F} \cap C_0(\Omega_2)$, we see that

$$h_k := (f_k)_+ \wedge (g_k)_+ \in \mathcal{F} \cap C_0(\Omega_1 \cap \Omega_2) \subset \mathcal{F}(\Omega_1 \cap \Omega_2).$$

Setting $u_k = (f_k)_+ - (g_k)_+$ and noticing that $u_k \xrightarrow{\mathcal{F}} 0$ as $k \to \infty$, we obtain by Proposition(2) that $|u_k| \xrightarrow{\mathcal{F}} 0$ as $k \to \infty$. It follows that

$$h_k = \frac{1}{2}[(f_k)_+ + (g_k)_+ - |(f_k)_+ - (g_k)_+|] \xrightarrow{\mathcal{F}} f \text{ as } k \to \infty.$$

Since $\mathcal{F}(\Omega_1 \cap \Omega_2)$ is a closed and, hence, weakly closed subspace of $\mathcal{F}$, we conclude that $f \in \mathcal{F}(\Omega_1 \cap \Omega_2)$. For a signed function $f \in \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_2)$, we have $f_+, f_- \in \mathcal{F}(\Omega_1) \cap \mathcal{F}(\Omega_1)$, whence, by the first part of the proof, $f_+, f_- \in \mathcal{F}(\Omega_1 \cap \Omega_2)$ and $f = f_+ - f_- \in \mathcal{F}(\Omega_1 \cap \Omega_2)$, which finishes the proof.

Proposition(7) *Let U be a non-empty open subset of M, and let $u \in \mathcal{F}$ such that* $\sup(u) \subset U$ *and is compact. Then* $u \in \mathcal{F}(U)$.

Proof: We can assume that $u \geq 0$ because a signed u follows from the decomposition $u = u_+ - u_-$. Next, we can assume that u is bounded because otherwise consider a sequence $u_k := u \wedge k$ that tends to u in $\mathcal{F}$ -norm as $k \to \infty$ by Proposition (2) ; if we already know that $u_k \in \mathcal{F}(U)$ then we can conclude that also $u \in \mathcal{F}(U)$. Hence, we can assume in the sequel that u is non-negative and bounded in *M*, say $0 \leq u \leq 1$.

Let φbe a cut-off function for the pair $(\operatorname{supp}(u), U)$. Let $\{u_k\}_{k=1}^{\infty}$ be a sequence from $\mathcal{F} \cap C_0(M)$ such that $u_k \overset{\mathcal{F}}{\rightarrow} u$ as $k \to \infty$. As $u \geq 0$, we have by the last results in Proposition(3) that $(u_k)_+ \overset{\mathcal{F}}{\rightarrow} u$ as $k \to \infty$ and $|(u_k)_+ - \varphi| \overset{\mathcal{F}}{\rightarrow} |u - \varphi|$ as $k \to \infty$. It follows that

$$(u_k)_+ \wedge \varphi = \frac{1}{2}[(u_k)_+ + \varphi - |(u_k)_+ - \varphi|] \overset{\mathcal{F}}{\rightarrow} \frac{1}{2}[u + \varphi - |u - \varphi|] = u \wedge \varphi = u \quad \text{as } k \to \infty.$$

Since $(u_k)_+ \wedge \varphi \in \mathcal{F} \cap C_0(U)$, we conclude that $u \in \mathcal{F}(U)$.

Proposition(8). *Let Ω be a precompact open subset of M and U be an open subset of M, and let K be a closed subset of M such that $K \subset U$ (see Fig. 34). Let $u \in \mathcal{F}$ be a function such that $u_+ \in \mathcal{F}(\Omega)$ and $u \leq \psi$ in $\Omega \setminus K$ for some $0 \leq \psi \in \mathcal{F}$. Then*

$$(u - \psi)_+ \in \mathcal{F}(\Omega \cap U). \tag{11}$$

Proof: Since $u - \psi \leq u_+ \in \mathcal{F}(\Omega)$, it follows by Proposition(2) that $(u - \psi)_+ \in \mathcal{F}(\Omega)$. Let us verify that

$$(u - \psi)_+ \in \mathcal{F}(\Omega), \tag{12}$$

which will then imply (11) by Proposition(6) Indeed noticing that $(u - \psi)_+ = 0$ in $\Omega \setminus K$ and in Ω^c , we see that

$$\operatorname{supp}((u - \psi)_+) \subset \overline{K \cap \Omega} \subset K \cap \overline{\Omega}.$$

On the other hand, the set $K \cap \overline{\Omega}$ is compact and is contained in *U*, so that (12) follows from Proposition (7).

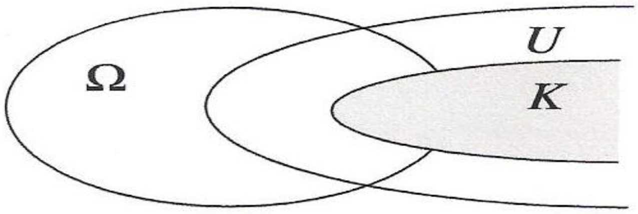

Figure 3: Domains Ω . U and K.

The next theorem is the basic technical result.

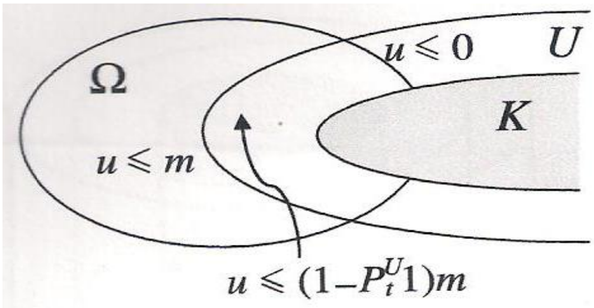

Figure 4: Illustration in the classical case

$$u \leq 0 \text{ on } \partial\Omega \cap U \ \big(\text{instead } of \ u_+ \in \mathcal{F}(\Omega)\big)$$

$$u \leq m \text{ on } \partial U \cap \Omega \text{ for some } m \geq 0 \ (\text{instead of } u \leq m \text{ on } \Omega \setminus K).$$

$$u(t,\cdot) \to 0 \text{ as } t \to 0 \text{ in } \Omega \cap U.$$

then $u \leq (1 - P_t^U \mathbf{1}_U)m$ in $(0, T_0) \times (\Omega \cap U)$ (see Fig 4). Indeed, the function $v = (1 - P_t^U \mathbf{1})m$ satisfies the heat equation in $(0, \infty) \times U$, the boundary conditions $v \geq 0$ on $\partial\Omega, v = m$ on ∂U,, and the initial condition $v(t,\cdot) \to 0$ as $t \to 0$ in *U*. Applying the classical parabolic maximum principle in $\Omega \cap U$, we obtain $u \leq v$.

Corollary (9). For Ω_n, Ω_{n+1} , $n \geq 1$, *be two sequence of non-empty open subsets of M*. Then

$$\mathcal{F}(\Omega_n) \cap \mathcal{F}(\Omega_{n+1}) = \mathcal{F}(\Omega_n \cap \Omega_{n+1}).$$

Proof : Given $\mathcal{F}(\Omega_n \cap \Omega_{n+1}) \subset \mathcal{F}(\Omega_i)$ for $i = 1,2,3,\dots$ we give

$$\mathcal{F}(\Omega_n \cap \Omega_{n+1}) \subset \mathcal{F}(\Omega_n) \cap \mathcal{F}(\Omega_{n+1}).$$

For the opposite inclusion, we can show $f \in \mathcal{F}(\Omega_n) \cap \mathcal{F}(\Omega_{n+1})$ implies that $f \in \mathcal{F}(\Omega_n \cap \Omega_{n+1})$. We assume that $f \geq 0$. Now let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be two sequences from $\mathcal{F} \cap C_0(\Omega_n)$ and $\mathcal{F} \cap C_0(\Omega_{n+1})$, respectively such that $f_k \longrightarrow f$ and $g_k \longrightarrow f$ in $\mathcal{F}$ -norm. For $f \geq 0$ then $f_+ = f$, show that

$$(f_k)_+ \overset{\mathcal{F}}{\to} f \ \text{ and } \ (g_k)_+ \overset{\mathcal{F}}{\to} f \ \text{ as } \ k \to \infty.$$

Since $(f_k)_+ \in \mathcal{F} \cap C_0(\Omega_n)$ and $(g_k)_+ \in \mathcal{F} \cap C_0(\Omega_{n+1})$, we see that

$$h_k := (f_k)_+ \wedge (g_k)_+ \in \mathcal{F} \cap C_0(\Omega_n \cap \Omega_{n+1}) \subset \mathcal{F}(\Omega_n \cap \Omega_{n+1}).$$

Setting $u_k = (f_k)_+ - (g_k)_+$ for $k \to \infty$ we have $u_k \overset{\mathcal{F}}{\to} 0$ by using again that $|u_k| \overset{\mathcal{F}}{\to} 0$ as $k \to \infty$. Hence

$$h_k \overset{\mathcal{F}}{\to} f \ \text{ as } \ k \to \infty.$$

Since $\mathcal{F}(\Omega_n \cap \Omega_{n+1})$ is a closed and hence weakly closed subspace of $\mathcal{F}$, we see that $f \in \mathcal{F}(\Omega_n \cap \Omega_{n+1})$ .For the signed function give the result.

Theorem(10) *Let (M, d, μ) be a metric measure space and let ($\mathcal{E}, \mathcal{F}$) be a regular Dirich form in $L^2(M,\mu)$. Let $\Omega \subset M$ be a precompact open set and $U \subset M$ be an open such that $\mu(U) < \infty$. Let u be a weak subsolution of the heat equation in $(0, T_0) \times (\Omega \cap U)$ where $T_0 \in (0, +\infty]$, such that*

$$u_+(t,\cdot) \in \mathcal{F}(\Omega) \ for\ any\ t \in (0, T_0), \tag{13}$$

$$u_+(t,\cdot) \xrightarrow{L^2(\Omega\cap U)} 0 \ \ as\ t \to 0. \tag{14}$$

Let K be a closed subset of M such that $K \subset U$. Then, for any $t \in (0, T_0)$ and for almost all $x \in M$,

$$u(t,x) \leq \left(1 - P_t^U 1_U(x)\right) \sup_{0<s\leq t} \|u_+(s,\cdot)\|_{L^\infty(\Omega\setminus K)}, \tag{15}$$

provided that $\sup_{0<s\leq t} \|u_+(s,\cdot)\|_{L^\infty(\Omega\setminus K)} < \infty$.

Proof: Outside Ω the inequality (15) is trivial because $u \leq 0$ by (13). In $\Omega \setminus U$ (15) is also obvious because $P_t^U 1_U = 0$ and $K \subset U$. It remains to prove (15) in $\Omega \cap U$. Fix a number $T \in (0, T_0)$ and define m by

$$m = \sup_{0<t\leq T} \|u_+(t,\cdot)\|_{L^\infty(\Omega\setminus K)}. \tag{16}$$

Let us first prove that, for any $t \in (0, T)$ and for μ-almost all $x \in \Omega \cap U$,

$$u(t,x) \leq m. \tag{17}$$

Let ϕ be a cut-off function for the pair (Ω, M) and consider the function

$$w = u - m\phi. \tag{18}$$

Then (17) will follow if we show that $w \leq 0$ in $(0, T) \times (\Omega \cap U)$. The latter will be proved by using the maximum principle of that we need to verify the following conditions.

(a) The function w is a weak subsolution of the heat equation in $(0, T) \times (\Omega \cap U)$.

Indeed, the functionϕ, considered as a function of (t, x), is a weak supersolution of the heat equation in $(0,\infty) \times \Omega$, since for any non-negative function $\psi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(\phi,\psi) = \lim_{t\to 0} t^{-1}(\phi - P_t\phi, \psi) = \lim_{t\to 0} t^{-1}(1 - P_t\phi, \psi) \geq 0.$$

Since u is a weak subsolution in $(0, T) \times (\Omega \cap U)$, we see from (18) that so is w.

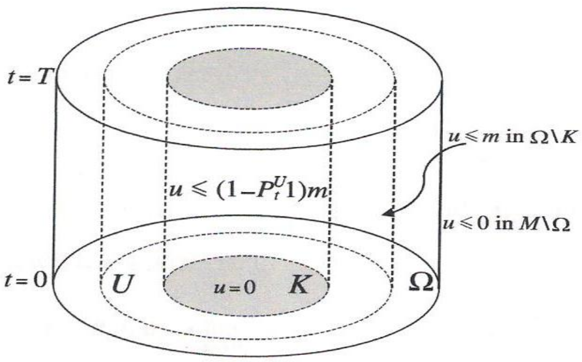

Figure 5: Illustration

in the case $U \subset \Omega$.

(i) For any $t \in (0,T)$, we have $w_+(t,\cdot) \in \mathcal{F}(\Omega \cap U)$. Indeed, using the facts that $u_+(t,\cdot) \in \mathcal{F}(\Omega)$ and $u \le m = m\phi$ in $\Omega \backslash K$ (which is true by (16)), we obtain from Proposition(8) that

$$w_+(t,\cdot) = (u(t,\cdot) - m\phi)_+ \in \mathcal{F}(\Omega \cap U).$$

(ii) The initial condition $w_+(t,\cdot) \xrightarrow{L^2(\Omega \cap U)} 0$ as $t \to 0$ follows from $w_+(t,\cdot) \le u_+(t,\cdot)$ and (14).

Therefore, by the parabolic maximum principle of Proposition(4), we conclude that $w \le 0$ in $(0, T) \times (\Omega \cap U)$, thus proving (17).

We are now in a position to prove the following improvement of (17):

$$u \le (1 - P_t^U 1_U)m \quad \text{in} \ (0,T) \times (\Omega \cap U) \tag{19}$$

(see Fig. 5 where the case $U \subset \Omega$ is shown). The path $t \mapsto u(t,\cdot)$ is weakly differentiable in $L^2(\Omega \cap U)$ and, hence, is strongly continuous in $L^2(\Omega \cap U)$ see [524]. The same applies to the path $t \to P_t^U 1_U$ so that the inequality (19) extends to $t = T$ by continuity. Hence, (19) implies (15). Consider the function

$$v = u - m\phi(1 - P_t^U \mathbf{1}_U), \tag{20}$$

where m and ϕ are the same as above. As $\mu(U) < \infty$, we have $\mathbf{1}_U \in L^2(U,\mu)$ and, hence, $P_t^U \mathbf{1}_U \in \mathcal{F}(U)$. We claim that v is a weak subsolution of the heat equation in $(0, T) \times (\Omega \cap U)$. Since u is a weak subsolution, it suffices to show that the function

$$f := \phi(1 - P_t^U \mathbf{1}_U)$$

is a weak supersolution in $(0, T) \times (\Omega \cap U)$. Since the both functions ϕ and $P_t^U \mathbf{1}_U$ belong to $L^\infty(M) \cap \mathcal{F}$, so does the product $\phi P_t^U \mathbf{1}_U$, whence

$$f = \phi - \phi P_t^U \mathbf{1}_U \in L^\infty(M) \cap \mathcal{F}.$$

For any $t, s \in (0,T)$, we have that in $\Omega \cap U$,

$$\begin{aligned} f - P_s f &= \phi(1 - P_t^U \mathbf{1}_U) - P_s\big(\phi(1 - P_t^U \mathbf{1}_U)\big) \\ &\ge (1 - P_t^U \mathbf{1}_U) - P_s(1 - P_t^U \mathbf{1}_U) \\ &= (1 - P_s 1) - P_t^U \mathbf{1}_U + P_s(P_t^U \mathbf{1}_U) \ge P_{t+s}^U \mathbf{1}_U - P_t^U \mathbf{1}_U, \end{aligned}$$

which yields that, for any $0 \le \psi \in \mathcal{F}(\Omega \cap U)$,

$$\mathcal{E}(f,\psi) = \lim_{s \to 0} \frac{1}{s}(f - P_s f, \psi) \ge \lim_{s \to 0} \frac{1}{s}(P_{t+s}^U 1_u - P_t^U 1_U, \psi) = \left(\frac{\partial}{\partial t} P_t^U 1_U, \psi\right).$$

On the other hand,

$$\left(\frac{\partial f}{\partial t}, \psi\right) = \left(-\phi \frac{\partial}{\partial t} P_t^U 1_U, \psi\right) = -\left(\frac{\partial}{\partial t} P_t^U 1_U, \psi\right).$$

Therefore,

$$\left(\frac{\partial f}{\partial t}, \psi\right) + \mathcal{E}(f,\psi) \ge 0,$$

showing that f is a weak supersolution. Hence, we have proved that v is a weak subsolution. Since $v \le u$, it follows from (14) that

$$v_+(t,\cdot) \xrightarrow{L^2(U \cap \Omega)} 0 \ \text{as} \ t \to 0.$$

It remains to verify the boundary condition: $v_+(t,\cdot) \in \mathcal{F}(\Omega \cap U)$ for any $t \in (0,T)$. Observe that

$$u - m\phi \le 0 \quad \text{in} \ M \tag{21}$$

because we have

(a) $u - m\phi \le 0$ in $M \setminus \Omega$ by (122),

(b) $u - m\phi = u - m \le 0$ in $\Omega \setminus U$ by (16),

(c) $u - m\phi = u - m \le 0$ in $\Omega \cap U$ by (17).

Using (21), we obtain that in M

$$v = u - m\phi(1 - P_t^U \mathbf{1}_U) \le m\phi P_t^U \mathbf{1}_U \le m P_t^U \mathbf{1}_U.$$

Since the function $P_t^U \mathbf{1}_U$ belongs to $\mathcal{F}(U)$, we conclude by using Proposition(3) that also $v_+ \in \mathcal{F}(U)$. On the other hand, we have

$$v = u - m\phi(1 - P_t^U \mathbf{1}_U) \le u \le u_+ \in \mathcal{F}(\Omega),$$

whence it follows that $v_+ \in \mathcal{F}(\Omega)$. Therefore, by Proposition(6) we obtain that $v_+ \in \mathcal{F}(U \cap \Omega)$, thus proving the boundary condition. Finally, we conclude by the maximum principle of Proposition(4) that $v \le 0$ in $(0, T) \times (\Omega \cap U)$, whence (19) follows.

Remark (11). The boundary condition (13) in Theorem(10) can be relaxed as follows:

$$u_+(t,\cdot) \in \mathcal{F}(\Omega) \text{ for any } t \in (0, T_0) \cap \mathbb{Q}, \tag{22}$$

provided one assumes in addition that

$$t \mapsto u(t,\cdot) \text{ is weakly continuousin } L^2(\Omega), \tag{23}$$

$$t \mapsto \mathcal{E}(u(t,\cdot)) \text{ is locally bounded}, \tag{24}$$

for $t \in (0, T_0)$. Under the hypotheses (22)-(24) , the inequality (15) can be replaced by a stronger one:

$$u(t,x) \le \left(1 - P_t^U 1_U(x)\right) \sup_{\substack{0<s\le t \\ s\in\mathbb{Q}}} \|u_+(s,\cdot)\|_{L^\infty(\Omega\setminus K)}. \tag{25}$$

The proof goes exactly as the above except that the supremum for defining the constant m in (16) is taken only over *rational* $t \in (0, T]$. Then we need to verify that the functions w and v, defined by (18), (20), respectively, satisfy the boundary condition (13) for all *real* $t \in (0, T)$ in order to be able to use the maximum principle of Proposition(4). Indeed, for any $t \in (0, T)$, let $\{t_k\}_{k=1}^{\infty}$ be a sequence of rationals such that $t_k \to t$ as $k \to \infty$. By (18) and (23), we have

$$w(t_k,\cdot) - w(t,\cdot) = u(t_k,\cdot) - u(t,\cdot) \xrightarrow{L^2(\Omega)} 0,$$

and thus

$$w_+(t_k,\cdot) \xrightarrow{L^2(\Omega)} w_+(t,\cdot).$$

By (24), $\mathcal{E}(w(t_k,\cdot))$ is bounded as $k \to \infty$.Hence, we obtain by Proposition(1) that

$$w_+(t_k,\cdot) \xrightarrow{\mathcal{F}} w_+(t,\cdot).$$

Since $w_+(t_k,\cdot) \in \mathcal{F}(\Omega)$ by (308), we conclude that $w_+(t,\cdot) \in \mathcal{F}(\Omega)$. Similarly, one has $v_+(t,\cdot) \in \mathcal{F}(\Omega)$ for all real $t \in (0, T)$.

The inequality (15) gives a rise to various interesting comparison inequalities for heat semigroups and heat kernels that will be presented below. Before that, let us state a useful particular case of Theorem(10) when $U \subset \Omega$ (cf. Fig. 5).

Corollary(12). *Let (M, d, μ) be a metric measure space and let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirich- form in* $L^2(M, \mu)$. *Let* $\Omega \subset M$ *be a precompact open set and U be an open subset of* Ω. *Let u be a weak subsolution of the heat equation in* $(0, T_0) \times U$ *where* $T_0 \in (0, +\infty]$, *such that*

$$u_+(t,\cdot) \in \mathcal{F}(\Omega) \ \ for\ any\ t \in (0, T_0),$$

$$u_+(t,\cdot) \xrightarrow{L^2(U)} 0 \ \ as\ t \to 0. \tag{26}$$

Then the conclusion of Theorem(12) holds for any compact subset K of U, any $t \in (0, T_0)$ *and almost all* $x \in M$.

We give various applications of Theorem(10) to the semigroup solutions, including a specific case of quasi-local Dirichlet form.

Proposition(13). *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form in* $L^2(M, \mu)$, *and let* Ω, *U be two non-empty open subsets of M such that* $\mu(U) < \infty$. *Let K be any closed subset of M such that* $K \subset U$. *Then, for any* $0 \le f \in L^2(\Omega)$,

$$P_t^\Omega f(x) - P_t^U f(x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|P_s^\Omega f\|_{L^\infty(\Omega\setminus K)}, \tag{27}$$

for all $t > 0$ *and almost all* $x \in M$.

Proof: Without loss of generality, assume that $0 \le f \in L^\infty(\Omega)$ (otherwise, apply (27) to the function $f_k = f \wedge k$ and then pass to the limit as $k \to \infty$). Let $\{\Omega_i\}$ be a sequence of precompact open subsets exhausting Ω. Consider the function

$$u(t,\cdot) := P_t^{\Omega_i} f - P_t^{\Omega_i \cap U} f$$

and we shall verify that u satisfies all the hypothesis of Theorem(10) with the sets Ω_i and U. Indeed, u is a weak subsolution of the heat equation in $(0, \infty) \times (\Omega_i \cap U)$ because so are $P_t^{\Omega_i} f$ and $P_t^{\Omega_i \cap U} f$. Next, $u(t,\cdot) \in \mathcal{F}(\Omega_i)$ because both $P_t^{\Omega_i} f$ and $P_t^{\Omega_i \cap U} f$ belong to $\mathcal{F}(\Omega_i)$. Since both $P_t^{\Omega_i} f$ and $P_t^{\Omega_i \cap U} f$ converge to f as $t \to 0$ in $L^2(\Omega_i \cap U)$, it follows that $u(t,\cdot) \xrightarrow{L^2(\Omega_i \cap U)} 0$ as $t \to 0$. By Theorem (10), we obtain that

$$P_t^{\Omega_i} f - P_t^{\Omega_i \cap U} f \leq (1 - P_t^U \mathbf{1}_U) \ \sup_{0<s\leq t} \| P_s^{\Omega_i} f - P_s^{\Omega_i \cap U} f \|_{L^\infty(\Omega_i \setminus K)}$$
$$\leq (1 - P_t^U \mathbf{1}_U) \ \sup_{0<s\leq t} \| P_s^{\Omega} f \|_{L^\infty(\Omega \setminus K)}.$$

Noticing that $P_t^{\Omega_i \cap U} f \leq P_t^U f$ and then passing to the limit as $i \to \infty$, we obtain (5), as desired.

Let us mention for comparison that the following inequality was proved in [355]:

$$P_t^{\Omega} f(x) - P_t^U f(x) \leq \ \sup_{0<s\leq t} \| P_s^{\Omega} f \|_{L^\infty(\Omega \setminus K)}. \tag{28}$$

Obviously, (27) is an improvement of (28). On the other hand, the estimate (28) was proved in [355] for arbitrary open set U without the hypotheses of the finiteness of its measure. For applications of (28) see [6].

Given an open set $U \subset M$ and non-negative number ρ, define the ρ *−neighborhood* U_ρ of U as follows:

$$U_\rho = \{x \in M : d(x, U) < \rho\} \quad \text{if } \rho > 0,$$
$$U_\rho = U \quad \text{if } \rho = 0,$$

where $d(x, U) = \inf_{y \in U} d(x, y)$.

Theorem (14). *Assume that* $(\mathcal{E}, \mathcal{F})$ *is a* ρ*-local regular Dirichlet form in* $L^2(M, \mu)$ *where* $\rho \geq 0$. *Let U be an open subset of M such that* U_ρ *is precompact, and let u be a weak subsolution of the heat equation in* $(0, T_0) \times U$ *where* $T_0 \in (0, +\infty]$. *Assume that, for any* $t \in (0, T_0)$, $u(t,\cdot) \in L^\infty(M)$ *and*

$$u_+(t,\cdot) \xrightarrow{L^2(U)} 0 \ \ as \ t \to 0. \tag{29}$$

Then for any compact subset K of U, for all $t \in (0, T_0)$, *and almost all* $x \in U_\rho$,

$$u(t, x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\leq t} \| u_+(s,\cdot) \|_{L^\infty(U_\rho \setminus K)}, \tag{30}$$

provided $\sup_{0<s\leq t} \| u_+(s,\cdot) \|_{L^\infty(U_\rho \setminus K)} < \infty$.

Proof: Since $P_t^U \mathbf{1}_U = 0$ outside U, the inequality (30) is trivially satisfied if $x \in U_\rho \setminus U$. Hence, it suffices to prove (30) for $x \in U$. Fix an open subset W of U such that $\overline{W} \subset U$. Then $\overline{W_\rho} \subset U_\rho$ so that W_ρ is precompact. Let ϕ be a cut-off function for the pair (W_ρ, U_ρ). Let us show that the function $w = u\phi$ satisfies all the hypothesis of Corollary(12) where the domains Ω, U are replaced by U_ρ , W respectively. Note that the function u may not satisfy the condition (26) so that we have to use w instead.

Let us first show that w is a weak subsolution of the heat equation in $(0, T_0) \times W$. Indeed, since $u(t,\cdot), \phi \in \mathcal{F} \cap L^\infty(M)$ for any $t \in (0, T_0) \times W$, it follows that $w(t,\cdot) \in \mathcal{F}$. Since u is a subsolution in $(0, T_0) \times W$ and $\phi \equiv 1$ in W, we have, for any non-negative function $\psi \in \mathcal{F}(W)$,

$$\left(\frac{\partial w}{\partial t}, \psi\right) = \left(\phi \frac{\partial u}{\partial t}, \psi\right) = \left(\frac{\partial u}{\partial t}, \psi\right) \leq -\mathcal{E}(u, \psi)$$
$$= -\mathcal{E}(w, \psi) + \mathcal{E}\big((\phi - 1)u, \psi\big) = -\mathcal{E}(w, \psi). \tag{31}$$

where we have used the fact that $\mathcal{E}\big((\phi - 1)u, \psi\big) = 0$ by the ρ-locality of $\mathcal{E}$, because $\mathrm{supp}(\psi) \subset \overline{W}$, and the function $(\phi - 1)u$ is compactly supported outside $\overline{W_\rho}$, so that the distance between the supports of ψ and $(\phi - 1)u$ is larger than ρ.

Since $\mathrm{supp}\varphi \subset U_\rho$, we see that $\mathrm{supp} w(t,\cdot) \subset U_\rho$, and hence, $w(t,\cdot) \in \mathcal{F}(U_\rho)$ and, $w_+(t,\cdot) \in \mathcal{F}(U_\rho)$. Moreover, it follows from (29) that

$$w_+(t,\cdot) = \phi u_+(t,\cdot) \xrightarrow{L^2(W)} 0 \ \text{ as } t \to 0.$$

Hence, w satisfied the required boundary and initial conditions, and by Corollary (11) we obtain that in $(0, T_0) \times W$,

$$u(t, x) = w(t, x) \leq \left(1 - P_t^W \mathbf{1}_W(x)\right) \sup_{0<s\leq t} \| w_+(s,\cdot) \|_{L^\infty(U_\rho \setminus K)}$$
$$\leq \left(1 - P_t^W \mathbf{1}_W(x)\right) \sup_{0<s\leq t} \| u_+(s,\cdot) \|_{L^\infty(U_\rho \setminus K)}.$$

Taking an exhaustion of U by sets like W and then passing to the limit as $W \to U$, we obtain (30).

For the case of local Dirichlet forms, we obtain the following improvement of Theorem(14) where the condition of the compactness of U_ρ is dropped.

Corollary (15). *Assume that* $(\mathcal{E}, \mathcal{F})$ *is a local regular Dirichlet form in* $L^2(M, \mu)$. *Let U be an open subset of M and let u be a weak subsolution of the heat equation in* $(0, T_0) \times U$ *where* $T_0 \in (0, +\infty]$. *Assume that, for any* $t \in (0, T_0)$, *the function* $u(t,\cdot) \in L^\infty(M)$ *and*

$$u_+(t,\cdot) \xrightarrow{L^2(U)} 0 \ \ as\ t \to 0.$$

Then, for any compact subset K of U, for all $t \in (0, T_0)$, *and almost all* $x \in U$,

$$u(t,x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|u_+(s,\cdot)\|_{L^\infty(U\setminus K)}, \qquad (32)$$

$provided$ $\sup_{0<s\le t}\|u_+(s,\cdot)\|_{L^\infty(U\setminus K)} < \infty$

As an another consequence of Theorem(14), we obtain the following useful comparison inequality for heat semigroups.

Corollary(16). *Assume that* $(\mathcal{E},\mathcal{F})$ *is a* ρ*-local regular Dirichlet form in* $L^2(M,\mu)$ *where* $\rho \ge 0$. *Let U,* Ω *be two open subsets of M such that* U_ρ *is precompact and* $U_\rho \subset \Omega$. *Then for any* $0 \le f \in L^\infty(M)$, *for all* $t > 0$ *and almost all* $x \in U_\rho$,

$$P_t^\Omega f(x) - P_t^U f(x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|P_s^\Omega f\|_{L^\infty(U_\rho\setminus K)} \qquad (33)$$

for any compact subset K of U.

Moreover, if $\rho = 0$, *that is,* $(\mathcal{E},\mathcal{F})$ *is local then the same is true without assuming that* U_ρ *is precompact. In this case, (33) becomes*

$$P_t^\Omega f(x) - P_t^U f(x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|P_s^\Omega f\|_{L^\infty(U\setminus K)}. \qquad (34)$$

Proof: Consider the function

$$u(t,\cdot) = P_t^\Omega f(\cdot) - P_t^U f(\cdot),$$

that is bounded on M for any $t > 0$, is a weak subsolution of the heat equation in $(0,\infty) \times U$, and satisfies the initial condition (29). Hence, it follows from (30) that, for all $t > 0$ and almost all $x \in U_\rho$,

$$P_t^\Omega f(x) - P_t^U f(x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|P_s^\Omega f - P_s^U f\|_{L^\infty(U_\rho\setminus K)},$$

whence (33) follows.

In the case of a local form, one passes from precompact U to arbitrary U as in the proof of Corollary(15).

The inequality (33) can be improved as follows:

$$P_t^\Omega f(x) - P_t^U f(x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{\substack{0<s\le t\\ s\in\mathbb{Q}}} \|P_s^\Omega f\|_{L^\infty(U_\rho\setminus K)}, \qquad (35)$$

because the function $u = P_t^\Omega f - P_t^U f$ automatically satisfies conditions (23) and (24). Since $U \subset \Omega$, it suffices to verify that the function $u = P_t^\Omega f$ satisfies (23) and (24).Indeed, (23) follows from the strong continuity of the semigroup $\{P_t^\Omega\}$ in $L^2(\Omega)$ whilst (24) follows from the fact that $\mathcal{E}(P_t^\Omega f)$ is a decreasing function of t, the latter being a consequence of the identity

$$\mathcal{E}(P_t^\Omega f) = \int_0^\infty \lambda e^{-2\lambda t} d(E_\lambda f, f),$$

where $\{E_\lambda\}$ is the spectral resolution of the operator Δ_Ω , the generator of $\left(\mathcal{E},\mathcal{F}(\Omega)\right)$. Hence, (35) follows from

$$u(t,x) \le \left(1 - P_t^U(x)\right) sup_{\substack{0<s\le t\\ s\in\mathbb{Q}}} \|u_+(s,.)\|_{L^\infty(U_\rho\setminus K)}$$

The estimate (34) with $K = \emptyset$ was proved also in [6]. A useful particular case of (34) is when the function f vanishes in U. In this case, (33) becomes

$$P_t^\Omega f(x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|P_s^\Omega f\|_{L^\infty(U_\rho\setminus K)}. \qquad (36)$$

We will also prove a symmetric comparison inequality for the heat kernel of a ρ-local Dirichlet form .The motivation is as follows. Let $(\mathcal{E},\mathcal{F})$ be an arbitrary regular Dirichlet form and let U, $V \subset \Omega$ be three open subsets of M such that

$U \cap V = \emptyset$. We claim that, for all $t, s > 0$ and μ-almost all $x \in U, y \in V$,

$$P_{t+s}^\Omega(x,y) \le [1 - P_t^U \mathbf{1}_U(x)] \|P_s^\Omega(\cdot,y)\|_{L^\infty(\Omega\setminus U)}$$
$$+[1 - P_s^V \mathbf{1}_V(y)] \left\|p_t^\Omega(\cdot,x)\right\|_{L^\infty(\Omega\setminus V)}. \qquad (37)$$

Indeed, noticing that

$$\int_{\Omega\setminus U} p_t^\Omega(x,z) d\mu(z) \le 1 - P_t^\Omega \mathbf{1}_U(x) \le 1 - P_t^U \mathbf{1}_U(x),$$

we obtain that

$$\int_{\Omega\setminus U} p_t^{\Omega}(x,z)d\mu(z) \le \|p_s^{\Omega}(\cdot,y)\|_{L^\infty(\Omega\setminus U)} \int_{\Omega\setminus U} p_t^{\Omega}(x,z)d\mu(z)$$
$$\le [1-P_t^U 1_U(x)]\|p_s^{\Omega}(\cdot,y)\|_{L^\infty(\Omega\setminus U)} \quad (38)$$

In a similar way, we have

$$\int_{\Omega\setminus V} p_t^{\Omega}(x,z)p_s^{\Omega}(z,y)d\mu(z) \le [1-P_s^V \mathbf{1}_V(y)]\left\|p_t^{\Omega}(\cdot,x)\right\|_{L^\infty(\Omega\setminus V)}. \quad (39)$$

Therefore, by the semigroup property,

$$p_{t+s}^{\Omega}(x,y) = \int_{\Omega} p_t^{\Omega}(x,z)p_s^{\Omega}(z,y)d\mu(z)$$
$$\le \int_{\Omega\setminus U} p_t^{\Omega}(x,z)p_s^{\Omega}(z,y)d\mu(z) + \int_{\Omega\setminus V} p_t^{\Omega}(x,z)p_s^{\Omega}(z,y)d\mu(z).$$

which together with (38) and (39) yields (37).

The purpose of the next theorem is to use the ρ-locality in order to replace in (37) the L^∞ -norms in $\Omega\setminus U$, $\Omega\setminus V$ by those in smaller sets, which is frequently critical for applications.

Theorem(17). *Let $(\mathcal{E},\mathcal{F})$ be a ρ-local regular Dirichlet form in $L^2(M,\mu)$ where $\rho\ge 0$, and let U, V, Ω be three open subsets of M such that U_ρ , V_ρ are precompact and $U_\rho, V_\rho \subset \Omega$. Assume that all the Dirichlet heat kernels $p_t^U, p_t^V, p_t^{\Omega}$ exist and that $p_t^{\Omega}(x,y)$ is locally bounded in $\mathbb{R}_+\times\Omega\times\Omega$. Then, for all $t,s>0$ and μ-almost all $x\in U, y\in V$,*

$$p_{t+s}^{\Omega}(x,y) \le \int_{\Omega} p_t^U(x,z)p_s^V(z,y)d\mu(z) + [1-P_t^U 1_U(x)] \sup_{s<t'\le t+s}\left\|p_{t'}^{\Omega}(\cdot,y)\right\|_{L^\infty(U_\rho\setminus K_1)}$$
$$+[1-P_s^V 1_V(y)] \sup_{t<t' t+s}\left\|p_{t'}^{\Omega}(\cdot,x)\right\|_{L^\infty(V_\rho\setminus K_2)}, \quad (40)$$

where K_1, K_2 are any compact subsets of U and V respectively.

In the case ρ = 0, that is, when $(\mathcal{E},\mathcal{F})$ is local, the assumption of the compactness of U_ρ, V_ρ can be dropped.

Proof: Let v be a non-negative function from $L^\infty\cap L^1(V)$. Setting $f=P_s^{\Omega}v$ and noticing that all the hypotheses of Corollary(16) are satisfied, we obtain by (35) that the following inequality is true in *U* for all *t* > *0*:

$$P_{t+s}^{\Omega}v \le P_t^U(P_s^{\Omega}v) + [1-P_t^U\mathbf{1}_U] \sup_{\substack{0<t'\le t\\ t'\in\mathbb{Q}}}\left\|P_{t'+s}^{\Omega}v\right\|_{L^\infty(U_\rho\setminus K_1)}$$
$$= P_t^U(P_s^{\Omega}v) + [1-P_t^U\mathbf{1}_U] \sup_{\substack{s<t'\le t+s\\ t'\in\mathbb{Q}}}\left\|P_{t'}^{\Omega}v\right\|_{L^\infty(U_1\setminus K_1)}, \quad (41)$$

where we have used that $P_t^{\Omega}f = P_{t+s}^{\Omega}v$. Consider the function

$$F(y) := \sup_{\substack{s<t'\le t+s\\ t'\in\mathbb{Q}}} \operatorname{esup}_{z\in U_\rho\setminus K_1} p_{t'}^{\Omega}(z,y),$$

which is bounded in V. Note that $F(y)$ is measurable as the supremum of a countable family of measurable functions of *y* since

$$y\mapsto \operatorname{esup}_{z\in U_\rho\setminus K_1} p_{t'}^{\Omega}(z,y)$$

is measurable *t* varies in $\mathbb{Q}$. We have then

$$\sup_{\substack{s<t'\le t+s\\ t'\in\mathbb{Q}}}\left\|P_{t'}^{\Omega}v\right\|_{L^\infty(U_\rho\setminus K_1)} = \sup_{\substack{s<t'\le t+s\\ t'\in\mathbb{Q}}} \operatorname{esup}_{z\in U_\rho\setminus K_1}\int_V p_{t'}^{\Omega}(z,y)v(y)d\mu(y)$$
$$\le \int_V F(y)v(y)d\mu(y). \quad (42)$$

Multiplying (41) by a non-negative function $u\in L^\infty\cap L^1(U)$ and integrating over *U*, we obtain

$$(P_{t+s}^{\Omega}v,u) \le (P_t^U(P_s^{\Omega}v),u) + \iint_{U\times V}[1-P_t^U 1_U(x)]F(y)u(x)v(y)d\mu(x)d\mu(y). \quad (43)$$

On the other hand, observe that

$$(P_t^U(P_s^{\Omega}v),u) = (P_s^{\Omega}v, P_t^U u) = (v, P_s^{\Omega}P_t^U u). \quad (44)$$

Using (35) again, now with $f=P_t^U u$ and with *V* in place of *U*, we obtain the following inequality in *V*:

$$P_s^{\Omega}P_t^U u = P_s^{\Omega}f \le P_s^V f + [1-P_s^V 1_V] \sup_{\substack{0<t'\le s\\ t'\in\mathbb{Q}}}\left\|P_{t'}^{\Omega}f\right\|_{L^\infty(V_\rho\setminus K_2)}. \quad (45)$$

Observing that $P_t^U u \le P_t^{\Omega}u$, we obtain that

$$P_{t'}^{\Omega}f = P_{t'}^{\Omega}P_t^U u \le P_{t'}^{\Omega}P_t^{\Omega}u = P_{t'+t}^{\Omega}u.$$

Similarly to (42), we have

$$\sup_{\substack{t<t'\le t+s\\ t'\in\mathbb{Q}}}\left\|P_{t'}^{\Omega}u\right\|_{L^\infty(V_\rho\setminus K_2)} \le \int_U G(x)u(x)d\mu(x)$$

where

$$G(x) := \sup_{\substack{t<t' \leq t+s \\ t' \in \mathbb{Q}}} \operatorname{esup}_{z\in V_\rho \setminus K_2} p_{t'}^{\Omega}(z,x)$$

is a bounded measurable function on U. Substituting into (45), we obtain in V

$$P_t^{\Omega} P_t^U u \leq P_s^V(P_t^U u) + [1 - P_s^V \mathbf{1}_V] \int_V G(x)u(x)d\mu(x). \qquad (46)$$

Multiplying (46) by v and integrating over V, we obtain

$$(v, P_s^{\Omega} P_t^U u) \leq \left(v, P_s^V(P_t^U u)\right) + \iint_{U\times V} [1 - P_s^V 1_V(y)]\, G(x)u(x)v(x)d\mu(x)d\mu(x).$$

Combining this with (43) and (154), we obtain

$$(P_{t+s}^{\Omega} v, u) \leq \left(v, P_s^V(P_t^U u)\right)$$
$$+\iint_{U\times V} [1 - P_t^U 1_U(x)]F(y)u(x)v(y)d\mu(x)d\mu(y)$$
$$+\iint_{U\times V} [1 - P_s^V 1_V(y)]G(x)u(x)v(y)d\mu(x)d\mu(y).$$

Since

$$\left(v, P_s^V(P_t^U u)\right) = \iint_{U\times V} \left(\int_{\Omega} p_t^U(x,z) p_s^V(z,y)d\mu(z)\right) u(x)v(y)d\mu(x)d\mu(y),$$

we can rewrite the previous inequality in the form

$$\iint_{U\times V} p_{t+s}^{\Omega}(x,y)u(x)v(y)d\mu(x)d\mu(y) \leq \iint_{U\times V} \Phi(x,y)u(x)v(y)d\mu(x)d\mu(y), \quad (47)$$

where

$$\Phi(x,y) = \int_{U\cap V} p_t^U(x,z)p_s^V(z,y)d\mu(z) + [1 - P_t^U \mathbf{1}_U(x)]F(y) + [1 - P_s^V \mathbf{1}_V(y)]G(x).$$

Obviously, $\Phi(x,y)$ is a bounded measurable function on $U \times V$. By [6], the inequality (47) implies

$$P_{t+s}^{\Omega}(x,y) \leq \Phi(x,y)$$

for almost all $x \in U$ and $y \in V$, which proves (40).

In the case of a local form ($\mathcal{E},\mathcal{F}$), one obtains the claim for arbitrary open sets U, V by passing to the limit when exhausting U and V by precompact open sets.

We introduce a technique for self-improvement of pointwise upper estimates of the heat kernel of a local, conservative, regular Dirichlet form. This issue was addressed in [7, 2, 5,3] on abstract metric measure spaces, and in [8, 9, 10] on some fractal sets. Motivated by the application of symmetric comparison inequalities for the heat kernels in [1], we here present an alternative approach to such results, which is based on Theorem(17)

Let $\{P_t\}_{t\geq 0}, \{P_t^{\Omega}\}_{t\geq 0}$ be the semigroups of the Dirichlet forms $(\mathcal{E},\mathcal{F}), \left(\mathcal{E},\mathcal{F}(\Omega)\right)$ respectively as before. For any $x \in M$ and $r > 0$, define the metric ball

$$B(x,r) = \{y \in M: d(x,y) < r\}.$$

For any ball $B = B(x,r)$ and any positive constant λ, denote by λB the ball $B(x,\lambda r)$.

Recall that a Dirichlet form ($\mathcal{E},\mathcal{F}$) in $L^2(M,\mu)$ is called *conservative* if the heat semigroup $\{P_t\}_{t\geq 0}$ *of* $(\mathcal{E},\mathcal{F})$ satisfies the following property:

$$P_t 1 = 1 \text{ in } M \text{ for any } t > 0.$$

Lemma(18). *Assume that* $(\mathcal{E},\mathcal{F})$ *is a conservative, regular Dirichlet form in* $L^2(M,\mu)$*, and let* $\{P_t\}_{t\geq 0}$ *be the heat semigroup of* $(\mathcal{E},\mathcal{F})$*. Assume that* $\phi(r,t)$ *is a non-negative function on* $(0,\infty)\times(0,)$ *such that* $\phi(r,\cdot)$ *is increasing in* $(0,\infty)$ *for every r > 0. If, for any t > 0 and any ball B in M of radius r,*

$$P_t \mathbf{1}_{B^c} \leq \phi(r,t) \;\; in \; \frac{1}{4}B, \qquad (48)$$

Then

$$1 - P_t^B \mathbf{1}_B \leq 2\phi\left(\frac{r}{4},t\right) \;\; in \; \frac{1}{4}B. \qquad (49)$$

Proof : Applying the estimate (37) with $\Omega = M$, $U = B$, $K = \frac{3}{4}\bar{B}$ and $P_t 1 = 1$, we obtain that, for any $t > 0$ and almost everywhere in M,

$$P_t^B \mathbf{1}_{\frac{1}{2}B} \geq P_t \mathbf{1}_{\frac{1}{2}B} - \sup_{0<s\leq t} \left\| P_s \mathbf{1}_{\frac{1}{2}B} \right\|_{L^\infty\left(\left(\frac{1}{4}\bar{B}\right)^c\right)}. \qquad (50)$$

For any $x \in \frac{1}{4}B$, we have that $B(x,r/4) \subset \frac{1}{2}B$ (see Fig. 6). Using the identity $P_t 1 = 1$, we have that, for any

$x \in \frac{1}{4}B$,

$$P_t \mathbf{1}_{\frac{1}{2}B} = 1 - P_t \mathbf{1}_{\left(\frac{1}{2}B\right)^c} \geq 1 - P_t 1_{B(x,r/4)^c}.$$

Applying (48) for the ball $B(x, r/4)$, we see that

$$P_t \mathbf{1}_{B(x,r/4)^c} \leq \phi(r/4, t) \text{ in } B(x, r/16).$$

It follows that, for any $x \in \frac{1}{4}B$,

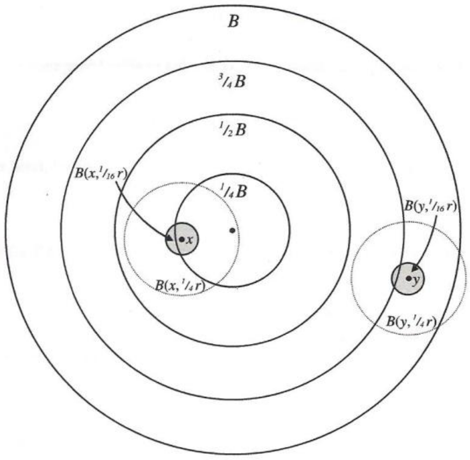

Figure 6: Illustration to the proof of Lemma (18).

$$P_t \mathbf{1}_{\frac{1}{2}B} \geq 1 - \phi(r/4, t) \quad \text{in } B(x, r/16).$$

Covering $\frac{1}{4}B$ by a countable family of balls $B(x_k, r/16)$ where $x_k \in \frac{1}{4}B$, we obtain that

$$P_t \mathbf{1}_{\frac{1}{2}B} \geq 1 - \phi(r/4, t) \text{ in } \frac{1}{4}B. \tag{51}$$

On the other hand, for any $y \in \left(\frac{3}{4}\bar{B}\right)^c$, we have that $\frac{1}{2}B \subset B(y, r/4)^c$, and so

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq P_s 1_{B(y,r/4)^c}.$$

Applying (48) for the ball $B(y, r/4)$ at time s and using the monotonicity of $\phi(r, s)$ in s, we obtain that, for any $0 < s \leq t$,

$$P_s \mathbf{1}_{B(y,r/4)^c} \leq \phi(r/4, s) \leq \phi(r/4, t) \text{ in } B(y, r/16).$$

It follows that, for any $y \in \left(\frac{3}{4}\bar{B}\right)^c$ and any $0 < s \leq t$,

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \phi(r/4, t) \text{ in } B(y, r/16),$$

which implies that

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \phi(r/4, t) \text{ in } \left(\frac{3}{4}\bar{B}\right)^c. \tag{52}$$

Combining (50), (51) and (52), we obtain that, for any $t > 0$,

$$P_t^B \mathbf{1}_B \geq P_t^B \mathbf{1}_{\frac{1}{2}B} \geq 1 - 2\phi(r/4, t) \text{ in } \frac{1}{4}B, \tag{53}$$

which was to be proved.

In the next statement, we use a function $F: M \times M \times (0, \infty) \to (0, \infty)$ with the following properties:

(F1): $F(x, y, s) = F(y, x, s)$ for all $x, y \in M$ and $s > 0$;

(F2): $F(x, y, s)$ is decreasing in s for any $x, y \in M$;

(F3): there exist $\alpha, C > 0$ such that

$$\frac{F(z, y, s)}{F(x, y, s)} \leq C \left(1 + \frac{d(x, z)}{s}\right)^{\alpha} \tag{54}$$

for all $x, y, z \in M$ and $s > 0$.

Theorem(19). *Let $(\mathcal{E}, \mathcal{F})$ be a conservative, local, regular Dirichlet form in $L^2(M, \mu)$. Let h be a positive* *increasing function on* $(0,+\infty)$. *Assume in addition that the following two conditions hold:*

(i) *The heat kernel* p_t *of* $(\mathcal{E},\mathcal{F})$ *exists and satisfies the inequality*

$$p_t(x,y) \le F(x,y,h(t)), \tag{55}$$

for all t > 0, μ-almost all $x,y \in M$, *where F is a function that satisfies the conditions* (F1)-(F3) *above.*

(ii) *There exist* $\varepsilon \in \left(0,\frac{1}{2}\right)$, *1 and δ > 0 such that, for any ball B of radius r > 0 and for any t > 0, we have*

$$P_t 1_{B^c} \le \varepsilon \ \ in \ \ \frac{1}{4}B \tag{56}$$

whenever $h(t) \le \delta r$.

Then, for all λ, t > 0 and μ-almost all $x,y \in M$,

$$p_t(x,y) \le CF\left(x,y,h\left(\frac{t}{2}\right)\right)\exp\left(-c' t\Psi\left(\frac{cr}{t}\right)\right) \tag{57}$$

where $r = d(x,y)$, *the constant* $C > 0$, *and Ψ is defined by*

$$\Psi(s) = \sup_{\lambda>0}\left\{\frac{s}{h(1/\lambda)} - \lambda\right\}. \tag{58}$$

Proof: Fix $t > 0$, two distinct points $x_0, y_0 \in M$ and set $r = \frac{1}{2}d(x_0,y_0)$. With $U = B(x_0,r), V = B(y_0,r), \Omega = M$ and $\rho = 0$, we obtain that, for μ-almost all $x \in B(x_0,r)$ and $y \in B(x_0,r)$,

$$p_t(x,y) \le \left[1 - P^U_{t/2}\mathbf{1}_U(x)\right] \sup_{t/2<s\le t} \operatorname{esup}_{z\in B(x_0,r)} p_s(z,y) \tag{59}$$

$$+\left[1 - P^V_{t/2}\mathbf{1}_V(y)\right] \sup_{t/2<s\le t} \operatorname{esup}_{z\in B(y_0,r)} p_t(z,x). \tag{60}$$

In what follows, we estimate the term on the right-hand side of (59), while the term in (60) can be treated similarly. We claim that, for all $\lambda > 0$,

$$1 - P^U_{t/2}\mathbf{1}_U \le C\exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right) \ \text{ in } \frac{1}{4}U. \tag{61}$$

Indeed, we see from (56) that the hypothesis (48) of Lemma(18) is satisfied with

$$\phi(r,t) = \begin{cases} \varepsilon, & \text{if } h(t) \le \delta r, \\ 1 & \text{otherwise.}\end{cases}$$

Therefore, by Lemma(18), we obtain that, for all balls B of radius r,

$$1 - P^B_t\mathbf{1}_B \le 2\phi\left(\frac{r}{4},t\right) \le 2\varepsilon \ \text{ in } \frac{1}{4}B,$$

provided that $h(t) \le \delta r/4$. It follows from [5] (see also [6]) that, for any ball B of radius r and for any $\lambda > 0$,

$$P_t 1_{B^c} \le C\exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right) \ \text{ in } \frac{1}{2}B.$$

Using Lemma(18) again, this time with the function

$$\phi(r,t) = C\exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right),$$

We obtain

$$1 - P^B_t 1_B \le 2C\exp\left(c'\lambda t - \frac{cr/4}{h(1/\lambda)}\right) \ \text{ in } \frac{1}{4}B,$$

which proves (61).

On the other hand, for all $z \in B(x_0,r)$ and $x \in B(x_0,r)$, we have that $z \in B(x,2r)$, whence by condition (F3)

$$\frac{F(z,y,h(t/2))}{F(x,y,h(t/2))} \le C\left(1+\frac{2r}{h(t/2)}\right)^\alpha \le 2^\alpha C\left(1+\frac{r}{h(t/2)}\right)^\alpha.$$

Noting that h is increasing and $F(x,y,\cdot)$ is decreasing, we have from (55) that, for all $\frac{1}{2} \le s \le t$ and for μ-almost all $z \in B(x_0,r)$ and $y \in B(y_0,r)$,

$$p_s(z,y) \le F(z,y,h(s)) \le F(z,y,h(t/2))$$

$$= F(x,y,h(t/2))\frac{F(z,y,h(t/2))}{F(x,y,h(t/2))} \le 2^\alpha CF(x,y,h(t/2))\left(1+\frac{r}{h(t/2)}\right)^\alpha.$$

Therefore, we have, for almost all $y \in B(y_0, r)$,

$$\sup_{t/2<s\leq t} \operatorname{esup}_{z\in B(x_0,r)} p_s(z,y) \in CF\big(x,y,h(t/2)\big)\left(1+\frac{r}{h(t/2)}\right)^{\alpha}. \quad (62)$$

Combining (61) and (62) and a similar estimate for the term in (60), we obtain from (59) and (60) that, for μ-almost all $x \in B\left(x_0, \frac{1}{4}r\right), y \in \left(y_0, \frac{1}{4}r\right)$,

$$p_t(x,y) \leq CF\big(x,y,h(t/2)\big)\left(1+\frac{r}{h(t/2)}\right)^{\alpha} \exp\left(c'\lambda t - \frac{cr}{h(1/\lambda)}\right). \quad (63)$$

In order to absorb the middle term to the exponential on the right-hand side in (63), fix r, t and consider the function

$$G(\lambda) := \frac{cr}{h(1/\lambda)} - c'\lambda t,$$

where c' , c are the same as in (63). Using this with $\lambda = 2/t$ and the elementary inequality

$$\alpha \log\left(1+\frac{r}{h(t/2)}\right) \leq \frac{1}{2}\frac{cr}{h(t/2)} + c''$$

$$= \frac{1}{2}G(2/t) + c' + c'' \leq \frac{1}{2}\sup_{\lambda>0} G(\lambda) + c' + c''.$$

Therefore,

$$\left(1+\frac{r}{h(t/2)}\right)^{\alpha} \exp\big(-\sup_{\lambda>0} G(\lambda)\big) \leq \exp\left(-\frac{1}{2}\sup_{\lambda>0} G(\lambda) + c' + c''\right)$$

$$\leq C \exp\left(-\frac{1}{2}\sup_{\lambda>0} G(\lambda)\right)$$

$$\leq C \exp\left(-\frac{1}{2}G(\lambda)\right).$$

Therefore, we obtain from (63) that, for any $\lambda > 0$ and μ-almost all $x \in B\left(x_0, \frac{1}{4}r\right), y \in B\left(y_0, \frac{1}{4}r\right)$.

$$p_t(x,y) \leq CF\big(x,y,h(t/2)\big) exp\left(-\frac{1}{2}G(\lambda)\right). \quad (64)$$

Since $M \times M \setminus \text{diag}$ can be covered by a countable family of sets $B\left(x_0, \frac{1}{4}r\right) \times B\left(y_0, \frac{1}{4}r\right)$ as above, it follows that (64) holds for μ-almost all $x, y \in M$. Taking sup in $\lambda > 0$, we obtain (57).

Let us give an example to illustrate Theorem(19), Set

$$V(x,r) := \mu\big(B(x,r)\big)$$

and assume in the sequel that the following volume doubling condition ($V\,D$) is satisfied: there is a constant $C_D \geq 1$ such that

$$V(x,2r) \leq C_D V(x,r) \quad (65)$$

for all $x \in M$ and $r > 0$. It is known that ($V\,D$) implies the existence of a constant $\alpha > 0$ such that

$$\frac{V(x,R)}{V(y,r)} \leq C_D\left(\frac{d(x,y)+R}{r}\right)^{\alpha} \quad (66)$$

for all $x, y \in M$ and $0 < r \leq R$ (see [6]).

Define functions h and F as follows:

$$h(t) = t^{1/\beta}$$

and

$$F(x,y,s) = \frac{C}{\sqrt{V\big(x,h(s)\big)V\big(y,h(s)\big)}},$$

for all $t, s > 0$ and $x, y \in M$, where $\beta > 1$ is some constant. It follows from (66) that F satisfies conditions (F1)-(F3). It is easy to see that the supremum in (58) is attained at $\lambda = cs^{\frac{\beta}{\beta-1}}$ so that

$$\Psi(s) = cs^{\frac{\beta}{\beta-1}}.$$

The estimate (57) becomes

$$p_t(x,y) \le \frac{C}{\sqrt{V(x,t^{1/\beta})V(y,t^{1/\beta})}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right).$$

for all $t > 0$ and almost all $x, y \in M$. Using (66) again and applying the same argument as in the proof of Theorem(19), we obtain that

$$p_t(x,y) \le \frac{C}{V(x,t^{1/\beta})} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right). \tag{67}$$

In particular, if $V(x,r) \simeq r^\alpha$ for some $\alpha > 0$, then (67) becomes

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right). \tag{68}$$

Proposition(20). *Let $F(x,y)$ be a non-negative μ-measurable function of $x, y \in M$. Then the function*

$$f(x) = \operatorname{esup}_y F(x,y)$$

Is measurable.

Proof: Fix a pointwise realization of F. Assume first that F is bounded. For any $x \in M$, consider the mapping

$$L^1 \ni \varphi \longmapsto T\varphi(x) := \int_M F(x,y)\varphi(y)d\mu(y)$$

which is a bounded linear functional on L^1 . We have

$$f(x) = \sup_{\|\varphi\|_1 \le 1} T\varphi(x).$$

Since T is continuous in φ, the supremum can be replaced by the one over a dense subset $S \subset L^1$, that is,

$$f(x) = \sup_{\|\varphi\|_1 \le 1, \varphi \in S} T\varphi(x).$$

Since $T\varphi$ is a measurable function, the supremum over a countable family is also measurable, and hence, the function f is measurable.

For an arbitrary F, consider $F_k = F \wedge k$, we have from above that $f_k(x) := esup_y F_k(x,y)$ is measurable. Note that the sequence $\{f_k\}_{k=1}^\infty$ increases and converges to f pointwise as $k \to \infty$. Hence, the function f is measurable.

Corollary(21). *Let $\sum_{j=1}^\infty F(x_j, y)$ be a series of non-negative μ-measurable functions of $x_j, y \in M$, $j \ge 1$. Then the series of functions* [11].

$$f(x_j) = \operatorname{esup}_y \sum_{j=1}^\infty F(x_j, y)$$

Is measurable.

Proof: If F is bounded. Then for any $x_j \in M$, we consider the mapping

$\varphi \longmapsto T\varphi(x_j)$ such that $\varphi \in L^1$

and

$$T\varphi(x_j) := \int_M \sum_{j=1}^\infty F(x_j, y)\varphi(y)d\mu(y)$$

which are a bounded linear functionals on L^1 . We have

$$f(x_j) = \sup_{\|\varphi\|_1 \le 1} \sum_{j=1}^\infty T\varphi(x_j).$$

Since T is continuous in φ, the supremum can be replaced by the one over a dense subset $S \subset L^1$, that is,

$$\sum_{j=1}^\infty f(x_j) = \sup_{\|\varphi\|_1 \le 1, \varphi \in S} \sum_{j=1}^\infty T\varphi(x_j).$$

Since $\sum_{j=1}^\infty T\varphi(x_j)$ are measurable functions, then the supremum over a countable family is also measurable ,and hence, the functions f , x_j are measurable.

References

[1]. A. Grigor'yan, L. Saloff-Coste, (2009), Heat kernel on manifolds with ends, Ann. Inst. Fourier (Grenoble) 59.

[2]. J. Kigami, (2004), Local Nash inequality and inhomogeneous of heat kernels, Proc. Lond. Math. Soc. (89), 525–544.

[3]. A. Grigor'yan, J. Hu, (2008) Upper bounds of heat kernels on doubling spaces, preprint.

[4]. M. Fukushima, Y. Oshima, M. Takeda, (1994) Dirichlet Forms and Symmetric Markov Processes, De Gruyter Stud. Math., De Gruyter.

[5]. A. Grigor'yan, J. Hu, (2008) Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces, Invent. Math. (174) : 81–126.

[6]. A. Grigor'yan, J. Hu, K.-S. Lau,(2009), Heat kernels on metric spaces with doubling measure, in: Proceedings of Conference on Fractal Geometry in Greifswald IV, Birkhäuser, pp. 3–44.

[7]. M.T. Barlow, vol. 1690, Springer, 1998, Diffusions on Fractals, Lecture Notes in Math., pp. 1–121.

[8]. M.T. Barlow, R.F. Bass, (1999), Brownian motion and harmonic analysis on Sierpínski carpets, Canad. J. Math. (4) 51, 673–744.

[9]. B.M. Hambly, T. Kumagai, (1999), Transition density estimates for diffusion processes on post critically finite self-similar fractals, Proc. Lond. Math. Soc. (3) 79, 431–458.

[10]. Shawgy Hussein and Arabi Ali Aamir, (2015), Heat Kernel Estimates and Comparison Inequalities For Heat Semigroups on metric measure spaces", PhD , Bakht Alrada.