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Research Article

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A proof of projection method about classical iterative methods

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Abstract Gauss-Sediel and Successive Over Relaxation (SOR) are two classical iterative methods to solve the large linear system $Ax = b$. In this paper, we prove that the two iterative methods are two orthogonal projection methods.

Keywords orthogonal projection, Gauss-Sediel, SOR

1. Introduction

Given an $n \times n$ real matrix A, and a real n-vector b, we consider to find x belonging to R^n such that

$$Ax = b, \qquad (1.1)$$

where A is coefficient matrix and b is the right-hand side vector. Jacobi, Gauss-Seidel and SOR are three efficient methods suitable for solving the problem (1.1). They are all iterative methods by modifying one or a few components of an approximate vector solution at a time, and the criteria for modifying a component in order to improve an iterate is to annihilate some component of the residual vector $b - Ax$.

We split A into three parts

$$A = D - E - F$$

where D is the diagonal of A, $-E$ is the strict lower part and $-F$ its strict upper part. We let x_k be the k-th iterate. With the above notation, the Jacobi iteration in vector form can be written as

$$x_{k+1} = D^{-1}(E + F)x_k + D^{-1}b. \qquad (1.2)$$

Similarly, the Gauss-Seidel iteration in vector form can be written as

$$x_{k+1} = (D - E)^{-1}Fx_k + (D - E)^{-1}b. \qquad (1.3)$$

The difference between (1.2) and (1.3) is that the approximate solution of Gauss-Seidel is updated immediately after the new component is determined.

By introducing a parameter ω, the SOR iteration is based on the splitting

$$\omega A = (D - \omega E) - (\omega F + (1 - \omega)D),$$

and the SOR iteration in vector form can be written as

$$x_{k+1} = (D - \omega E)^{-1}[\omega F + (1 - \omega)D]x_k + (D - \omega E)^{-1}\omega b \qquad (1.4)$$

A lot of application about these three iterative methods can be found in [2-4]

2. Projection Method

Most of the existing practical iterative techniques for solving large linear systems of equations utilize a projection process in one way or another.

Let κ and L be two $m-$dimensional subspaces of R^n. In general, κ is called search subspace and L the subspace of constraints. A projection technique onto the subspace κ and orthogonal to L is a process which finds an approximate solution $\tilde{x}$ to (1.1) by imposing the conditions that $\tilde{x}$ belong to κ and that the new residual vector be orthogonal to L.

$$\text{Find } \tilde{x} \in \kappa, \text{ such that } b - A\tilde{x} \perp L.$$

If we exploit the knowledge of an initial guess x_0 to the solution, then the approximate problem should be refined as

$$\text{Find } \tilde{x} \in x_0 + \kappa, \text{ such that } b - A\tilde{x} \perp L.$$

Most standard techniques use a succession of such projections. Typically, a new projection step uses a new pair of subspaces κ and L, and an initial guess x_0 equal to the most recent approximation obtained from the previous projection step, see [1] for details.

Theorem 3.1 An elementary Gauss-Seidel step as defined by (1.3) is a projection step with $\kappa = L = span\{e_i\}$.

Proof We rewrite (1.4) as

$$(D - E)x_{k+1} = Fx_k + b,$$

from the perspective of component updates, this iterate process is actually a n steps update, namely n steps projection. We assume that x_{k+1} has two components and $x_{k+1}(i)$ is the $i-$th component of x_{k+1}. The update equation of $x_{k+1}(i)$ is

$$a_{i1}x_{k+1}(1) + a_{i2}x_{k+1}(2) + \cdots + a_{i(i-1)}x_{k+1}(i-1) + a_{ii}x_{k+1}(i)$$
$$= -a_{i(i+1)}x_k(i+1) - a_{i(i+2)}x_k(i+2) - \cdots - a_{in}x_n(k) + b_i.$$

This equation can be seen as the update from the vector

$$(x_{k+1}(1), x_{k+1}(2), \cdots, x_k(i), x_k(i+1), \cdots, x_k(n))$$

to the vector

$$(x_{k+1}(1), x_{k+1}(2), \cdots, x_{k+1}(i), x_k(i+1), \cdots, x_k(n)).$$

If i changes from 1 to n, all the components of the update is finished, i.e.

$$(x_{k+1}(1), x_{k+1}(2), \cdots, x_{k+1}(i), x_k(i+1), \cdots, x_k(n))$$
$$= (x_{k+1}(1), x_{k+1}(2), \cdots, x_k(i), x_k(i+1), \cdots, x_k(n)) + t(0,0,\cdots,0,1,0,\cdots,0)', \quad (2.1)$$

here $(A)'$ represents the transpose of the matrix A. We denote the second vector $(0,0,\cdots,0,1,0,\cdots,0)'$ on the right hand side of (2.1) as $e_{(i)}$, and the left vector on the left hand side as $x_{(k+1,i)}$. So (2.1) can be rewritten as

$$x_{(k+1,i)} = x_{(k+1,i-1)} + te_{(i)}.$$

The $i-$th component of the residual vector $b - Ax$ equal to zero, i.e.

$$(e_{(i)})'(b - Ax_{(k+1,i)}) = 0.$$

So Gauss-Seidel step is a projection step with $\kappa = L = span\{e_i\}$.

Theorem 3.2 An elementary SOR step as defined by (1.4) is a projection step with $\kappa = L = span\{e_i\}$.

Proof We rewrite (1.4) as

$$(D - \omega L)x_{k+1} = ((1-\omega)D + \omega U)x_k + \omega b,$$

from the perspective of component updates, this iterate process is actually a n steps update, namely n steps projection. We assume that x_{k+1} has two components and $x_{k+1}(i)$ is the $i-$th component of x_{k+1}. The update equation of $x_{k+1}(i)$ is

$$\omega a_{i1}x_{k+1}(1)+\omega a_{i2}x_{k+1}(2)+\cdots+\omega a_{i(i-1)}x_{k+1}(i-1)+a_{ii}x_{k+1}(i)$$
$$=(1-\omega)a_{ii}x_k(i)-\omega a_{i(i+1)}x_k(i+1)-a_{i(i+2)}x_k(i+2)-\cdots-\omega a_{in}x_n(k)+\omega b_i\,.$$

This equation can be seen as the update from the vector

$$(x_{k+1}(1),x_{k+1}(2),\cdots,x_k(i),x_k(i+1),\cdots,x_k(n))$$

to the vector

$$(x_{k+1}(1),x_{k+1}(2),\cdots,x_{k+1}(i),x_k(i+1),\cdots,x_k(n))\,.$$

If i changes from 1 to n, all the components of the update is finished, i.e.

$$(x_{k+1}(1),x_{k+1}(2),\cdots,x_{k+1}(i),x_k(i+1),\cdots,x_k(n))$$
$$=(x_{k+1}(1),x_{k+1}(2),\cdots,x_k(i),x_k(i+1),\cdots,x_k(n))+t(0,0,\cdots,0,1,0,\cdots,0)'\,, \qquad (2.2)$$

(2.2) can be rewritten as

$$x_{(k+1,i)}=x_{(k+1,i-1)}+te_{(i)}\,.$$

The $i-$th component of the residual vector $b-Ax$ equal to zero, i.e.

$$(e_{(i)})'(b-Ax_{(k+1,i)})=0\,.$$

So Gauss-Seidel step is a projection step with $\kappa=L=span\{e_i\}$.

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