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Exact Solutions of Quasielastic Problems of Linear Theory of Viscoelasticity and Nonlinear Theory Viscoelasticity for Mechanically Incompressible Bodies

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Abstract In the paper we cite theorems that by fulfilling the condition of mechanical incompressibility of a material reduce the problem of nonlinear theory of viscoelasticity with V. V.Moskvitin determining equations to the problem of physically nonlinear theory of elasticity of theonomical bodies. Under the noted condition the represented theorems allow to reduce the problem on linear theory of viscoelasticity to the appropriate problem of elasticity theory. The suggested theorems are illustrated on an example of problems.

Keywords Quasielastic Problems, Viscoelasticity, Linear Theory, Nonlinear Theory

1. Statement and solution of the general problem. Give the statement of a quasistatic problem of nonlinear theory of viscoelasticity with V.V. Moskvitin determining equations [1] for mechanically incompressible bodies

$$2G_0 e_{ij} = f(\sigma_+) s_{ij} + \int_0^t \Gamma(t-\tau) f(\sigma_+) s_{ij} d\tau; \;\; \theta = 0; \tag{1.1}$$

or

$$s_{ij}/2G_0 = \varphi(\varepsilon_+) e_{ij} - \int_0^t \Gamma(t-\tau) \varphi(\varepsilon_+) e_{ij} d\tau; \;\; \theta = 0; \tag{1.2}$$

and

$$\sigma_{ij,j} + F_i = 0; \;\; \sigma_{ij} l_j |_{S_\sigma} = R_i; \;\; u_i |_{S_u} = u_{oi}; \tag{1.3}$$

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \quad or \quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}. \tag{1.4}$$

Here $i, j, k, l = 1,2,3;$ $u_i, \varepsilon_{ij}, \sigma_{ij}$ are the components of permutations, deformation and stress, respectively; $e_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij};$ $s_{ij} = \sigma_{ij} - \sigma\delta_{ij};$ $\varepsilon = \varepsilon_{ij}\delta_{ij}/3;$ $\sigma = \sigma_{ij}\delta_{ij}/3;$ δ_{ij} are Kronecker symbols; $\varepsilon_+ = (2e_{ij}e_{ij}/3)^{1/2};$ $\sigma_+ = (3s_{ij}s_{ij}/2)^{1/2};$ $G_0 = const$ is an instantaneous shear modulus of a material; f, φ are the functions of nonlinearity of a material; $\Gamma(t)$ and $L(t)$ are mutually-resolvent kernels; F_i and R_i are volumetric and surface forces, respectively; u_{0i} are boundary permutations.

The following theorems hold.

Theorem 1. Problem (1.1), (1.3), (1.4) has the following solution.

$$u_i = u_i^{'} + \int_0^t \Gamma(t-\tau)u_i^{'} d\tau;\ \varepsilon_{ij} = \varepsilon_{ij}^{'} + \int_0^t \Gamma(t-\tau)\varepsilon_{ij}^{'} d\tau,\ \sigma_{ij} = \sigma_{ij}^{'}, \qquad (1.5)$$

where the quantities $u_i^{'}, \varepsilon_{ij}^{'}, \sigma_{ij}^{'}$ are the solutions of the following problem of the theory of nonlinear elasticity of mechanically incompressible bodies

$$2G_0 e_{ij}^{'} = f\left(\sigma_+^{'}\right)s_{ij}^{'};\ \theta' = 0; \qquad (1.6)$$

$$\sigma_{ij,j}^{'} + F_i = 0;\ \ \sigma_{ij}^{'} l_j\ |_{S_\sigma} = R_i;\ u_i^{'}\ |_{S_u} = u_{oi}^{'} \equiv u_{oi} - \int_0^t L(t-\tau)u_{0i} d\tau; \qquad (1.7)$$

$$\varepsilon_{ij}^{'} = \left(u_{i,j}^{'} + u_{j,i}^{'}\right)/2;\ \varepsilon_{ij,kl}^{'} + \varepsilon_{kl,ij}^{'} = \varepsilon_{ik,jl}^{'} + \varepsilon_{jl,ik}^{'}. \qquad (1.8)$$

Here we adopt the following denotation

$$e_{ij}^{'} = \varepsilon_{ij}^{'} - \varepsilon'\delta_{ij};\ s_{ij}^{'} = \sigma_{ij}^{'} - \sigma'\delta_{ij};\ \varepsilon_{ij}^{'} = \varepsilon_{ij}^{'}\delta_{ij}/3;\ \sigma' = \sigma_{ij}^{'}\delta_{ij}/3;\ \theta' = 3\varepsilon';$$

$$\varepsilon_+^{'} = \left(2e_{ij}^{'}e_{ij}^{'}/3\right)^{1/2};\varepsilon_+^{'} = \left(3s_{ij}^{'}s_{ij}^{'}/2\right)^{1/2}.$$

Theorem 2. Problem (1.2), (1.3), (1.4) has the following solution

$$u_i = u_i^{''};\ \varepsilon_{ij} = \varepsilon_{ij}^{''};\ \sigma_{ij}^{'} = \sigma_{ij}^{''} - \int_0^t L(t-\tau)\sigma_{ij}^{''} d\tau, \qquad (1.9)$$

where the quantities $u_i^{''}, \varepsilon_{ij}^{''}, \sigma_{ij}^{''}$ are the solutions of the following problem of the theory of nonlinear elasticity of mechanically incompressible bodies

$$s_{ij}^{''}/2G_0 = \varphi\left(\varepsilon_+^{''}\right)e_{ij}^{''};\ \theta^{''} = 0; \qquad (1.10)$$

$$\sigma_{ij}^{''} + F_i + \int_0^t \Gamma(t-\tau)F_i d\tau = 0; \qquad (1.11)$$

$$\sigma_{ij}^{''} l_j\ |_{S_\sigma} = R_i + \int_0^t \Gamma(t-\tau)R_i d\tau;\ \ u_i^{''}\ |_{S_u} = u_{0i}; \qquad (1.12)$$

$$\varepsilon_{ij}^{''} = \left(u_{i,j}^{''} + u_{j,i}^{''}\right)/2;\ \sigma_{ij,kl}^{''} + \varepsilon_{kl,ij}^{''} = \varepsilon_{ij,kl}^{''} + \varepsilon_{jl,ik}^{''}. \qquad (1.13)$$

Here we denote

$$e_{ij}^{''} = \varepsilon_{ij}^{''} - \varepsilon^{''}\delta_{ij};\ s_{ij}^{''} = \sigma_{ij}^{''} - \sigma^{''}\delta_{ij};\ \varepsilon^{''} = \varepsilon_{ij}^{''}\delta_{ij}/3;\ \sigma^{''} = \sigma_{ij}^{''}\delta_{ij}/3;\ \theta^{''} = 3\varepsilon^{''};$$

$$\varepsilon_+^{''} = \left(2e_{ij}^{''}e_{ij}^{''}/3\right)^{1/2};\ \sigma_+^{''} = \left(3s_{ij}^{''}s_{ij}^{''}/2\right)^{1/2}.$$

The proof of theorems 1 and 2 are carried out by direct substitution of formulae (1.5) and (1.9) into appropriate relations.

Theorems 1 and theorems 2 are also valid for $f = 1, \varphi = 1$ that holds in the case of the theory of linear viscoelasticity.

2. Examples. a) Pure bending of a straight beam. Accept that x_3 is an axis of a beam, x_1 and x_2 are principal central inertia axes of cross section whose area will be denoted by F. Move up x_1 in tension fibers, then x_2 will be a neutral axis. Lateral side of the beam is free from external forces and mass forces are absent. Let the moments $M(t)$ equal in size and apposite in sign and whose plane of action coincides with the plane x_1x_3 be applied on the ends of the beam.

Assume that beams material is mechanically incompressible and its properties are expressed by the laws of the theory of nonlinear viscoelasticity (1.1). And the problem on definition of components of stresses δ_{ij} and strains ε_{ij} is formed from relations (1.1), from the first relation of (1.3), the second relation of (1.4), the first boundary condition (1.3), where S_σ is taken as a lateral surface. Here $R_i = 0$. To these relations we add a boundary condition that should be satisfied on the ends of the beam

$$M(t) = \int_F \sigma_{33} x_1 dF \,. \tag{2.1}$$

Now, use formulae of (1.5). In this connection, relation (1.1), the first relation of (1.3), the second relation of (1.4) and the first boundary condition (1.3) are reduced to corresponding relations (1.6)-(1.8). Condition (2.1) is written in the following form:

$$M(t) = \int_F \sigma'_{33} x_1 dF \,. \tag{2.2}$$

Let for the material of the beam a non-linearity function $f(\sigma_+) = A\sigma_+^\alpha$ be found experimentally, i.e. the constants A and α are known.

Let's solve the problem composed of (1.6), the first and second relations of (1.7), the second relation of (1.8) and relation (2.2). represent the stress components σ'_{ij} in the form

$$\sigma'_{11} = \sigma'_{22} = \sigma'_{12} = \sigma'_{13} = \sigma'_{23} = 0; \;\; \sigma'_{33} = C(t) x_1^{1/(1+\alpha)} \tag{2.3}$$

where $C(t)$ is still unknown function, α is the known constants of the material.

Stress components satisfy the first two relations of (1.7) for $F_i = 0,\ R_i = 0$ that correspond to the conditions of our problem.

And satisfaction of condition (2.2) leads to the relation

$$M(t) = C(t) \int_F x^{\frac{2+\alpha}{1+\alpha}} dF \equiv C(t) I.$$

Hence $C(t) = M(t)/I$. Consequently,

$$\sigma'_{33} = \frac{M(t)}{I} x_1^{1/(1+\alpha)}. \tag{2.4}$$

It is easy to define $\sigma' = \sigma'_{33}/3;\ s'_{33} = 2\sigma'_{33}/3;\ \sigma'_+ = \sigma'_{33};\ s'_{11} = s'_{22} = -\sigma'_{33}/3;$ $s'_{12} = s'_{13} = 0$, where σ'_{33} is represented by formula (2.4). Allowing for these relations we find the strain tensor components ε'_{ij} by equations (1.6)

$$\varepsilon_{ij}^{'} = \frac{A}{3G_0}\frac{M^{1+\alpha}(t)}{I^{1+\alpha}}x_1; \;\; \varepsilon_{11}^{'} = \varepsilon_{22}^{'} = -\frac{1}{2}\varepsilon_{33}^{'}; \; \varepsilon_{12}^{'} = \varepsilon_{13}^{'} = \varepsilon_{23}^{'} = 0. \qquad (2.5)$$

It is easy to check that the found components $\varepsilon_{ij}^{'}$ identically satisfy six strain compatibility equations (1.8). After determination of $\sigma_{ij}^{'}$ and $\varepsilon_{ij}^{'}$ by formulae (1.5) we find the desired components of stresses σ_{ij} and strains ε_{ij} that arise in straight beam made of physically nonlinear viscoelastic material under pure bend:

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0; \;\; \sigma_{33} = \frac{M(t)}{I}x_1^{\frac{1}{1+\alpha}}; \qquad (2.6)$$

$$\varepsilon_{33} = \frac{Ax_1}{3G_0 I^{1+\alpha}}\left(M^{1+\alpha}(t) + \int_0^t \Gamma(t-\tau)M^{1+\alpha}(\tau)d\tau\right) \qquad (2.7)$$

$$\varepsilon_{11} = \varepsilon_{22} = -\frac{1}{2}\varepsilon_{33}; \; \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0. \qquad (2.8)$$

Solution of (2.6)-(2.8) coincides with the solution of the considered problem obtained in [2] in another way. It is also easy to verify that components of stress (2.6) and strain (2.7) and (2.8) satisfy all necessary equations and boundary conditions. Therefore, they are exact solutions of the considered nonlinear problem of viscoelasticity under conditions of power dependence of nonlinearity function and mechanical incompressibility of beam's material.

b) Plane deformation of a hollow thick walled cylinder by internal **pressure.** A hollow thick walled cylinder of internal radius a and external radius b is under the action of internal pressure $p(t)$. Mechanical properties of the cylinder's material are described by the equations of non-linear viscoelasticity of incompressible material (1.2). Radial stress σ_r, peripheral stress σ_φ, corresponding deformations $\varepsilon_r, \varepsilon_\varphi$ and permutation u (plane deformation) arising in the cylinder by (1.2), (1.3) and (1.4) satisfy the relations:

$$\frac{\sigma_\varphi - \sigma_r}{2G_0} = (\varepsilon_\varphi - \varepsilon_r)\varphi(\varepsilon_+) - \int_0^t R(t-\tau)(\varepsilon_\varphi - \varepsilon_r)\varphi(\varepsilon_+)d\tau; \;\; \theta = 0; \qquad (2.9)$$

$$\frac{\partial \sigma_r}{\partial r} = \frac{\sigma_\varphi - \sigma_r}{r}; \; \sigma_r\,|_{r=a} = -p(t), \;\; \sigma_r\,|_{r=b} = 0; \qquad (2.10)$$

$$\varepsilon_r = \frac{\partial u}{\partial r}, \; \varepsilon_\varphi = \frac{u}{r}; \; \varepsilon_z = 0. \qquad (2.11)$$

By formula (1.9) the problem (2.9)-(2.11) is reduced $\sigma_{33}^{'}$ the following problem:

$$\frac{\sigma_\varphi^{''} - \sigma_r^{''}}{2G_0} = (\varepsilon_\varphi^{''} - \varepsilon_r^{''})\varphi(\varepsilon_+^{''}); \;\; \theta^{''} = 0; \qquad (2.12)$$

$$\frac{\partial \sigma_r^{''}}{\partial r} = \frac{\sigma_\varphi^{''} - \sigma_r^{''}}{r}; \; \sigma_r^{''}\,|_{r=a} = -\left[p(t) + \int_0^t \Gamma(t-\tau)p(\tau)d\tau\right]; \;\; \sigma_r^{''}\,|_{r=b} = 0; \qquad (2.13)$$

$$\varepsilon_r^{''} = \frac{\partial u^{''}}{\partial r}; \;\; \varepsilon_\varphi^{''} = \frac{u^{''}}{r}; \;\; \varepsilon_z^{''} = 0. \qquad (2.14)$$

Problem (2.12)-(2.14) is a particular case of problem (1.10)-(1.13).

We represent the nonlinearity function $\varphi\left(\varepsilon_+''\right)$ where

$$\varepsilon_+'' = \frac{\sqrt{2}}{3}\left[\left(\varepsilon_r'' - \varepsilon_\varphi''\right)^2 + \varepsilon_\varphi''^2 + \varepsilon_r''^2\right]^{1/2}$$

in the form of the power function: $\varphi\left(\varepsilon_+''\right) = B\left(\varepsilon_+''\right)^\beta$. We use (2.14) from incompressibility condition $\theta'' = 0$ or $\varepsilon_r'' + \varepsilon_\varphi'' + \varepsilon_z'' = 0$ and get:

$$u'' = \frac{c(t)}{r},\ \varepsilon_r'' = -\frac{c(t)}{r^2},\ \varepsilon_\varphi'' = \frac{c(t)}{r^2},\ \varepsilon_+'' = \frac{2}{\sqrt{3}}\frac{c(t)}{r^2}, \qquad (2.15)$$

where $c(t)$ is a still unknown function. Allow for (2.15) in the first equation of (2.12) and determine the expression $\sigma_\varphi'' - \sigma_r''$ by the function $c(t)$. Using the obtained expression in the equilibrium equation (2.13) and a boundary condition for $r = a$ we determine the components of σ_r''

$$\sigma_r'' = \frac{2^{\beta+2} B G_0 c^{1+\beta}(t)}{3^{\beta/2}(2\beta+2)}\left(a^{-2\beta-2} - r^{-\beta-2}\right) - \left(p(t) + \int_0^t \Gamma(t-\tau)p(\tau)d\tau\right). \qquad (2.16)$$

We use boundary condition (2.13) for $r = b$, from (21.6) find the unknown function $c(t)$

$$c(t) = \left\{\frac{(2\beta+2)\left[p(t) + \int_0^t \Gamma(t-\tau)p(\tau)d\tau\right]}{\left(2/\sqrt{3}\right)^\beta 4BG_0\left(a^{-2\beta-2} - b^{-2\beta-2}\right)}\right\}^{1/(1+\beta)}. \qquad (2.17)$$

Allowing for (2.17) in (2.16) we determine the quantity σ_r'':

$$\sigma_r'' = \frac{b^{-2\beta-2} - r^{-2\beta-2}}{a^{-2\beta-2} - b^{-2\beta-2}}\left[p(t) + \int_0^t \Gamma(t-\tau)p(\tau)d\tau\right]. \qquad (2.18)$$

Now, using (2.15), (2.17) and (2.18), from the equation (2.9) we find σ_φ'':

$$\sigma_\varphi'' = \frac{(2\beta+1)r^{-2\beta-2} + b^{-2\beta-2}}{a^{-2\beta-2} - b^{-2\beta-2}}\left[p(t) + \int_0^t \Gamma(t-\tau)p(\tau)d\tau\right]. \qquad (2.19)$$

Permutation u'', deformations $\varepsilon_r'', \varepsilon_\varphi''$ are determined by formulae (2.15) with regard to (2.17). Consequently, we found exact analytic solution of problem (2.12)-(2.14). Now using formulae (1.9) we determine the desired solution of problem (2.9)-(2.11)

$$\sigma_r = \frac{b^{-2\beta-2} - r^{-2\beta-2}}{a^{-2\beta-2} - b^{-2\beta-2}}p(t);\quad \sigma_\varphi = \frac{(2\beta+1)r^{-2\beta-2} + b^{-2\beta-2}}{a^{-2\beta-2} - b^{-2\beta-2}}p(t). \qquad (2.20)$$

After determining σ_r and σ_φ, stress σ_z is found on the basis relations of plane deformation of incompressible material: $\sigma_z = \frac{1}{2}\left(\sigma_r + \sigma_\varphi\right)$. Strain components ε_φ and ε_r are determined on the basis of the second formula of (1.9)

$$\varepsilon_\varphi = -\varepsilon_r = \frac{1}{r^2}\left\{\frac{(2\beta+2)\left[p(t)+\int_0^t \Gamma(t-\tau)p(\tau)d\tau\right]}{\left(2/\sqrt{3}\right)^\beta 4BG_0\left(a^{-2\beta-2}-b^{-2\beta-2}\right)}\right\}^{\frac{1}{1+\beta}} . \qquad (2.21)$$

We can find permutation u from the second formula of (2.11): $u = \varepsilon_\varphi r$ where ε_φ is represented by the formula (2.21). The solution of (2.20)-(2.21) coincides with the solution of the considered problem obtained in [1] by another method. Alongside with this we can see by direct substitution that formulae (2.2)-(2.21) is the solution of problem (2.9)-(2.11).

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