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An Application of Backward Stochastic Equation (BSDE) in the Theory of Contingent Claim Valuation

Emmanuel Inyang[1], Bright O. Osu*[2], Chigozie Chibuisi[3]

[1]Research student in the Department of mathematics Abia state university Uturu
[2]Department of mathematics Michael Okpara University of Agriculture Umudike Abia State
[3]Department of Insurance, Faculty of Management Science, University of Jos, Plateau State, Nigeria

Abstract In this work we show the existence and uniqueness of Backward stochastic differential equation (BSDE) whose value is prescribed at the terminal time T. We proved under a certain technical condition, established and reformulated the comparison theorem for solutions of BSDE. Furthermore, based on the new approach, we briefly apply the result to determine the financial problem and behaviour of option price in the theory of contingent claim valuation $\xi \geq 0$ at maturity T, in a complete market, which pays an amount ξ at time T. Also, we predicted the replication claim with difference risk premium for long and short position. A study of replication with high interest rate for borrowing was carried out. Also, the hedge portfolio constructed by longing a contingent claim and shorting the unit was shown. In addition, pricing option problem of a contingent claim in a constrained case using viscosity solution was provided.

Keywords Backward stochastic differential equation, contingent claim, pricing and hedging portfolios, comparison theory, Malliavian derivative, viscosity solution

Introduction

Backward stochastic differential equations (BSDE) are new class of stochastic differential equation, whose value is prescribed at the terminal time T. BSDEs are one of the interesting areas with increasing activity because of their connection with economics, non-linear partial differential equation and mathematical finance. The solution of BSDE consists of two different processes, instead of one process.

$$-dY_t \ = -(\,t,Y_{t,}Z_t)\,d_t + Z_t dW_t, \qquad (1.1)$$

$$Y_T = \xi.$$

Where Y_t and Z_t are stochastic processes. W_t is a Brownian motion. ξ is a random variable called terminal condition and f is a given function. BSDEs were firstly defined and expressed linearly and become more popular after the general concept of BSDEs were considered by Paodoux and Peng in1990 [1], they proved the existence and uniqueness of continuous adapted processes as a pair (Y,Z) such that forgiven uniformly Lipschitz adapted stochastic processed-dimensional and square integrable terminal condition ξ. Financial option pricing, sometimes called Contingent claims are contracts whose outcomes depend on the evolution of one or more uncertain variables. Also, Contingent claim is another term of derivative with future payoff that is dependent on the realization. And the future payoff is contingent on the behaviours of some underlying assets, the theory of contingent claim is designed to measure risk and assign appropriate premium for risk. When pricing a financial option, we try to find a portfolio of stock that has the same value as the option and set price of the option equal to the price of this portfolio, called the replicating portfolio, however, not every financial option can be replicated with a portfolio of stock. These include option price, forward contract, interest rates and future contract. The value of forward contract is contingent claim and the forward contract is a special type of the derivative contract. The holder of a forward contract agrees to buy or sell the underlying asset at some delivery price K in the future. K is determined so that the cost of entering into the forward contract is zero at its inception. When pricing a financial option, this problem of pricing a contingent claim ξ in an incomplete market by Follmer and Schweizer [2] we show that this pricing rule corresponds to a standard valuation in a market where only the traded securities have a return different from the spot rate this price for ξ is still a solution of a linear BSDE. Recall that El-Karoui and Queneg [3] for the constrained case of an incomplete market for hedging problem with a higher interest rate for borrowing. The problem is to determine the option pricing of a contingent claim $\xi \geq 0$ of maturity T. Which is a contract pays an amount ξ at time Tin a complete market; it is possible to construct a portfolio which attains as final wealth the amount ξ. Therefore the corresponding BSDE given the dynamics of the value of the replication portfolio which is the fair price of contingent claim. In illustrate show the existence and uniqueness of square $-$integral solution of BSDE, in comparison the theorem can help prince a standard contingent claim ξ in a complete market. Suppose in general market there $n+1$asset, where the first one is riskless while the other is risky. The riskless asset prices S_s^0 satisfies

$$dS_s^0 = Y_s S_s^0 ds, \tag{1.2}$$

where $r(\cdot)$ is a non-dimensional, f_s $-$adaptated measurable process .The $i-th$ risky asset price$S_t^i, i = 1,2,3 \dots . n$ satisfies

$$dS_s^i = S_s^i [a_s^i dS + \sum_{i=1}^{n} \sigma^{ii} dB_s], \tag{1.3}$$

where $B = (B^i \dots \dots B^m)$ is a standard Brownian motion on$\mathbb{R}^n$,defined on probability space $(\Omega, \mathbb{R}, \mathcal{F})$ and where $a^i(\cdot)\sigma^{ii}(\cdot)$ are called the application rate and volatilityrate respectively and $\mathcal{F}_s$ adapted measurable processes. Moreover, the fair price is the market value of the hedging strategy, there exist some contingent claim ξ for which is empty. For such a contingent claim, the fair price is not defined. However, the upper price is for any square integrable non-negative contingent claim ξ.If $H(\xi)$ is nonempty and the upper price is well defined, then the market is said to be complete. Moreover, the fair price is the market value of hedging strategy in $H(\xi)$**.** The market is incomplete if there exist some contingent claim ξ for which $H(\varepsilon)$ is empty. And for such a contingent claim, fair price is not defined. Let us consider an agent with an initial endowment $X_0 \epsilon \mathbb{R}$, and investment horizon $T > 0$ and his allocation to the ithasset, for $i = 0,1,2..n$ be π_s^i, the corresponding number of shares be N_t^i and the total wealth be X_t at time$t$, it follows

$$X_s = \sum_{i=0}^{n} N_s^i S_s^i. \tag{1.4}$$

Suppose the agent strategy is self-financing, then we have:

$$dX_s = \sum_{i=0}^{n} N_s^i dS_s^i$$

$$\begin{aligned} Y_s = Y_0 + \int_0^T \pi\, d_s \int_o^T \sum_{i=1}^{n} \pi\, [\, a_s^i d_s + \sum_{i=1}^{n} \pi_s^{ij}\, dB_s^i] &= Y_s N_s^0 S_s^0 ds + \sum_{i=1}^{n} N_s^i S_s^i \big[a_s^i dS + \sum_{j=1}^{m} \sigma_s^{ii} dB_s^i\big] \\ &= Y_s\{[X - \sum_{i=1}^{n} \pi_s^i] + \sum_{i=1}^{n} a_s^i \pi_s^i\} + \sum_{j=1}^{m} \sum_{j=1}^{m} \sigma_s^{ii} \pi_s^i dB_s^i \\ &= [X_s X_s + \pi_s^*(a_s - Y_s 1)]ds + \pi_s^* \sigma_s dB_s^i \\ &= \big[[X_s X_s + \pi_s^*(a_s - Y_s 1)]dS + \pi_s^* \sigma_s dB_s\big]. \end{aligned} \tag{1.5}$$

This is called a wealth equation.

Assumptions

1. The appreciation rate $a = (a^1 \dots . . a^n)^*$ is a predictable and bounded process.
2. There exist a predictable and bounded valued process vector θ called a risk premium such that $a_s - Y_s 1 = \sigma_s \theta_s$
3. The non-negative risky rate is a predictable and bounded process
4. The volatility rate $\sigma = \sigma^{ii}$ is a predictable and bounded process σ_s and has full rank for all $s \in [0, S]$ and the inverse σ^{-1} has a bounded process

By this assumption, the market is arbitrage free and complete on $[0, S]$ and the wealth equationbecomes:

$$dX_s = [Y_s X_s + \sigma_s \theta_s]ds + \pi_s^* \sigma_s dB_s. \tag{1.6}$$

Definition 1.1.7: $\boldsymbol{\pi}(.)$is called an admissible portfolio if it is self-financing and $\pi(.) \mathcal{E}\, \mathbb{L}_s^2(\mathbb{R}^n)$.

Definition 1.1.8: A contingent claim ξ is settled at time T is a $\mathcal{F}_s$ – measurable random variable .The claim ξ is called replicable if there exist an initial X_0 and an admissible portfolio $\boldsymbol{\pi}(.)$, such that the corresponding $X(\cdot)$ satisfies $X(S) = \xi$

Definition 1.1.9: The contingent claim can be thought of as a contract which paysξ at maturity T. The arbitrage free pricing of a positive contingent claim is seen as the initial endowment and the inverse of the $n+1$ asset, the values of the portfolio at time T must be just enough to guarantee.

Definition 1.10: A market is called complete on $[0, T]$ if any claim $\xi \mathcal{E} L^2(\mathbb{R}^s)$ is replicable.

Definition 1.11: A self-financing trading strategy is a pair(X, σ^t, π), where X is the market value and $\pi = (\pi^i \dots . \pi^n)$ is the portfolio process, such that (X, σ^t, π^i) satisfies

$$dX_s = [Y_s X_s + \pi_s^* \sigma^t \theta^t] dS + \pi_S^* \sigma_S dB_S$$

$$= \int_0^s | \pi_s^* \sigma_s|^2 dS < +\infty. \qquad (1.7)$$

Here,X is the wealth process or the market value,π is the portfolio process, σ_s is the cumulative consumption process and σ_s is an increasing, right continuous.

2. Method

2.1. Comparison Theorem

The comparison theorem for BSDE is one of the classic and very important results of the properties of BSDE. It was first introduced by Peng [4] and Cao and Yam [5] under the Lipschitz hypothesis on the coefficient, with a special diffusion coefficient. The comparison theorem plays the same role that the maximum principle plays in the theory of partial differential equation in mathematical finance. It gives a sufficient condition for the wealth process to be nonnegative. Stochastic domination theorem plays an important role in the theory of stochastic processes as well as their application. The comparison theorem is as a result of the theory of BSDE existence and uniqueness path wise almost surly dominance. That is, when one process with probability one is greater than or equal to other.

2.2. Malliavin Derivative

Malliavin derivative is the notion of derivative appropriate to paths in wiener space, which are not differentiable in the usual sense. The Malliavian derivative is still the solution of linear BSDE. Then together with comparison theorem it can be applied to contingent claim option pricing. Consider the BSDE of the form

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t^* dB_t, Y_T = \xi, \qquad (2.1)$$

where $B = B_t \geq 0$ is Brownian motion in $\mathbb{R}^n (Y_{t,} t, \in [0, T]$ is a continuous $\mathbb{R}^d$ valued adapted process, $[Z_t \; t \in [0, T]$ is an $\mathbb{R}^{n \times d}$ valued predictable process and $\xi \in L^2(\mathbb{R}^d)$ and $(\mathcal{F}_t)_{t \geq 0}$ is the Brownian filtration equation (1.1) is equivalent to

$$dY_t = -f(t, Y_t Z_t)dt + \sum_{i=1}^n Z_t dB_t ,Y_t - \xi \qquad (2.2)$$

The differential equation (1.2) can be interpreted as the integral equation of the form

$$\xi - Y_t = -\int_t^T f(s, Y_s, Z_s)ds + \sum_{i=0}^n \int_t^T X_{-s} dB_s. \qquad (2.3)$$

By local martingale, we have$Y_T = \xi$, which implies$\xi = M_T - V_T$, $M_T = \mathbb{E}(\xi + V_T \mid f_T)$, and $Y_t = \mathbb{E}(\xi + \mid f_t) - V_{T,}$ for $t \in [t, T]$. Thus,$\xi - M_t + V_t = -\int_t^T f(s, Y_s, Z_s)ds + \sum_{i=0}^n \int_t^T Z_s dB_s$for any $t \in [t, T]$. Taking expectation on both sides gives;

$$\mathbb{E}(\xi \mid f_t) - M_t + V_t - E\left[\int_t^T f(s, Y_s, Z_s)ds \mid f_t \mid + \int_T^T f(s, Y_s, Z_s)ds\right],$$

$$V_t - V_t = \int_t^T f(s, Y_s, Z_s)ds , \qquad (2.4)$$

where Y and Z are considered as functions of V,

$$M_t = \mathbb{E}(\xi + V_t \mid f_t \mid) \text{ and } Y_t = \mathbb{E}(\xi + V_t \mid f_t \mid) - V_t. \qquad (2.5)$$

Z is determined uniquely by the martingale and the density process is given by

$M_t - M_t = \sum_{i=o}^n \int_t^T dw_s$, Y and$Z$ as Y_u and Z_u , so that $\frac{dV}{dt} = f(t, Y(u)_t, Z(u_t))$.

Proposition 2.1: Let (β, r) be a bounded ($\mathbb{R}\mathbb{R}$) valued predictable process,φ an element of $H_t^2(\mathbb{R})$ and ξ an element $\mathbb{L}_T^2(\mathbb{R})$.Then the LBSE

$$dY_t = [\varphi_t + Y_t \beta_t + Z_t^* r_t]dt + Z_t dW, \qquad Y_t = \xi$$

has a unique solution (Y, Z) in $\mathbb{K}^2(\mathbb{R}) \times H^2(\mathbb{R})^n$ and Y_t is given by

$Y_t = \frac{1}{T_t}\mathbb{E}\left[\xi T + \int_0^T Ts\varphi_s ds \mid f_t \mid\right],$

with $dT_t = T_t[B_t dt + \gamma_t^* dB_t], T_0 = 1.$

In particular, if ξ and φ are nonnegative, the process Y is the nonnegative if in addition $Y_{t=0}, \ as \ \varphi_{t=0}$

Proof: Let **β**, be bounded processes, then the linear generator

$f(t, Y_t, Z_t) = [\varphi_t + Y_t B_t + Z_t^* r_t]$

is uniformly Lipschitz, and there exist a unique square integrable solution (Y, Z) of the linear BSDE

$Y_t = \frac{1}{T_t} = \mathbb{E}\left[\xi T_t + \int_t^T T_s \varphi_s ds \mid f_t \mid\right],$

which satisfies

$Y_t = \frac{1}{T_t}\mathbb{E} \mid \xi T_t \mid = \xi.$

For terminal condition

$$Y_t = \frac{1}{T_t}\mathbb{E}\left[\xi T_t + \int_t^T T_s \varphi_s ds \parallel f_t \mid\right.$$

$dY_t = -[\varphi_t + Y_t\beta_t + Z_t^*\gamma_t)dt + Z_t^* dw.$

Now by the Ito lemma

$$\begin{aligned} d(X_t) &= Y_t dT_t + T_t dY_t + (Y.T) + T_t\varphi_t dt \\ &= Y_t[T_t(B_t dt + \gamma_t^* dW_t)] + T_t[-(\varphi_t + Y_t B_t + Z_t^*\gamma_t)dt + Z_t^* dW_t] + Z_t^* T_t \gamma_t^* dt + T_t \varphi_t dt \\ &= Y_t T_t V_t dW_t + T_t Z_t^* dW_t = (Y_t T_t V_t^* + T_t Z_t^*) dW_t. \end{aligned}$$

It implies X_t is local martingale, since $sup_{s \leq T} \mid Y_s \mid$ and $sup_{s \leq T} \parallel Ts \mid$ belong to $\mathbb{L}_T^2$

$X_T = Y_t T_t + \int_0^T T_s \varphi_s dS.$

We have $Y_t T_t + \int_0^T T_s\varphi_s ds \parallel f_t \mid = \mathbb{E}\left[\xi T_t + \int_0^T T_s \varphi_s ds\right] f_t \mid$, where Y_t satisfies

$dY_t = -[\varphi_t + Y_t\beta_t + Z_t^*\gamma_t]dt + Z_t^* dW,$

$T_t = \exp\left[\int_0^T B_s ds + \int_0^T \gamma_t^* dws - \frac{1}{2}\int_0^T \mid \gamma_s \mid^2 ds\right],$

which is a non-negative process.

Therefore, the nonnegative variable $\xi T_t + \int_0^T T_s \varphi_s ds \ has \ o \ expectation, and \ Y = O,$ if ξ and **φ** are nonnegative, Y_t is also non-negative and

$Y_0 = 0 = \mathbb{E}\left[\xi T_t + \int_0^T T_s \varphi_s ds\right].$

If $\xi = 0, \varphi_t = 0$ a.s., It follows that $Y_t = 0.$

Theorem 2.1 (Comparison Theorem): Considers the following BSDE;

$$dY_s = -f(s, Y_s, Z_s)\, ds - Z_s^* dW_S, Y_s = \xi \quad (2.6)$$

and

$$d\overline{Y_S} = -\overline{f}(s, \overline{Y}_s, \overline{Z}_s)dS - Z_s\overline{d}W_s, \overline{Y}_s = \overline{\xi} \quad (2.7)$$

On a filtered probability space (Ω, f, f_s, P) which satisfies the condition; (Ω, f, P) is a complete space, $(f_s)_t \geq 0$ is continuous all martingales on (Ω, f, f_s, P) is continuous. $Z\ (s, S): \mathbb{R}^d) \rightarrow \mathbb{H}^2(s, \overline{S})$; $\boldsymbol{R}^{nxd}$) is a prescribed mapping on valued in $\mathbb{H}^2(s, S)\boldsymbol{R}^{nd}$that f satisfies the Lipschitz condition and. Z satisfies the Lipschitz condition. The local−in−time property and the differential property to ensure the existence and uniqueness of solution. Let (Y, Z) and $(\overline{Y}, \overline{Z}) \in C^2(\boldsymbol{R})$ be the unique adapted solution (2.6) and (2.7) respectively. Assume that for any $\overline{Z}_1, \overline{Z}_2 C^2(\boldsymbol{R})$ and $t \in [0, S]$

$[(\overline{Z}_1 - \overline{Z}_2) - (\overline{Z}_1 - \overline{Z}_2)_S] \geq \alpha\ \mathbb{E}[\ |(\overline{Z}_1)_s - (\overline{Z}_2)_S|^2 ds]$ where α is positive constant.

(1) If $\xi \leq \overline{\xi}$ a.s, $f(s, \overline{Y}_s, \overline{Z}_s) \leq \overline{f}(s, Y_s, Z_s)$ a.s., then $Y_s \leq \overline{Y}_s$, a.s,$\forall, o \leq s \leq S.$

(2) If $\xi \geq \overline{\xi}$a.s, $f(s, \overline{Y}_s, \overline{Z}_s) \geq \overline{f}(s, \overline{Y}_s, \overline{Z}_s)$ a.s., then $Y_s \geq \overline{Y}_s$,a.s,$\forall\ o \leq s \leq S.$

Proof: Let $Y_s^* = Y_s - \overline{Y}_s, \xi^* = \xi - \overline{\xi}, Z_s^* = Z_s - \overline{Z}_s.$

From conditions 1 and 2 we obtain.

$Y_s^* = \xi^* + \int_s^S [f(s,\overline{Y}_s,\overline{Z}_s) - \overline{f}(s,\overline{Y}_s,\overline{Z}_s)] dS - \int_S^S dB_S.$

and

$Y_s^* = Y_s^* - \int_o^s (\overline{f}(s,Y_s,Z_s) - \overline{f}(s,\overline{Y}_s,\overline{Z}_s)\, dS + \int_o^s dB_s.$

$Y_s^*$is a continuous semi martingale it follows for $s\boldsymbol{\epsilon}(o,s)$.refer to (Cao Yan1999) it follows for $s\,\boldsymbol{\epsilon}[\,\mathbf{0},\boldsymbol{s}]$

$Y_s^{*+2} = \xi^{*+2} + 2\int_s^S Y_s^* + [(s,Y_s,Z_s) - \overline{f}(s,\overline{Y}_s,\overline{Z}_s) ds - 2\int_s^S Y^{*+2}\, dB - \int_s^S 1\,(Y_S > o) dB_s.$

Therefore we have:

$\left|Y_s^{*+2}\right| + \int_o^s 1\,(Y_S > o) dB_s = \left|\xi^*\right|^2 + 2\int_s^S Y_s^* + [(s,Y_s,Z_s) - \overline{f}(s,\overline{Y}_s,\overline{Z}_s) ds - 2\int_0^S \left|Y_s^*\right| dB_{s,}$

That $\int_s^S \left|Y_s\right| dB_s$ is a martingale by Burkholder –Davis–Gundy inequality

$$[\sup 0 \le s \le S \left| \int_0^S \left|Y_s^*\right| dB_{s,}\right] \le C\mathbb{E}\,[\,(\int_0^s \left|Y_s^*\right|^2 dB_s)^{\frac{1}{2}}\,]$$
$$\le C\mathbb{E}[\sup 0 \le s \le S \left|Y_s^*\right| (\int_0^s 1\,(dB_s)^{\frac{1}{2}}\,]$$
$$\le C\mathbb{E}[\sup 0 \le s \le S \left|Y_s^*\right| \left|Z_s\right|^{\frac{1}{2}}\,]$$
$$= \le \frac{c}{2}[\,[\sup 0 \le s \le S \left|Y_s^*\right|^2 \left| + \mathbb{E}\right| Z_s \left|\,\right] < \infty$$

C is a position constant, hence $\int_s^S dZ_s$ is martingale.Taking expectation on both sides we have

$$\mathbb{E}\left|Y_s^*\right|^2 + \mathbb{E}\,(\int_o^s 1\,(Y_s > o) dZ_s) = 2\mathbb{E}\,[\,Y_s^*[\,f(s,Y_s,Z_s) - \overline{f}(s,\overline{Y}_s,\overline{Z}_s) ds]$$
$$\le \mathbb{E}\,[\,\int_s^S \frac{2c^2}{\alpha}\left|Y_s^*\right|^2 + 1\,(Y_s > o)\frac{\alpha}{2c^2}\left|f(s,Y_s,Z_s) - \overline{f}(s,\overline{Y}_s,\overline{Z}_s)\right|^2 ds]$$
$$\le \mathbb{E}\int_s^S \frac{2c^2}{\alpha} Y_s^{*2} + 1\,(Y_s > o)\frac{\alpha}{2c^2}[\,c_2(\left|Y_s\right| + \left|Z_s\right| - \left|Z_s\right|^2]\,ds$$
$$\le \mathbb{E}\int_s^S \frac{2c^2}{\alpha} Y_s^{*2} + 1\,(Y_s > o)\,\alpha[\left|Y_s\right|^2 + \left|Z_s\right| - \left|Z_s\right|^2]\,ds.$$

where c^2 isthe Lipschitz constant and

$\mathbb{E}\int_s^S d\left|Z_s\right| \ge \alpha\mathbb{E}\,[\int_s^S \left|Z_s\right| - \left|Z_s\right|^2]\,ds$

implies

$$\mathbb{E}\,(Y_s^2) \le \mathbb{E}\,[\int_s^S \frac{2c^2}{\alpha} Y_s^2 + \alpha\left|Y_s^*\right|^2] ds \le \mathbb{E}\int_s^S (\frac{2c^2}{\alpha} + \alpha)\, Y_s^2 ds \le (\frac{2c^2}{\alpha} + \alpha)\int_s^S \mathbb{E}\,(Y_s^{*2}) ds.$$

Let$f\,(t) = \mathbb{E}\,(Y_{S-S}^2)$, $0 \le s \le S$, we get

$$f\,(t) \le (\frac{2c^2}{\alpha} + \alpha)\int_{S-s}^S \mathbb{E}\,(Y_s^{*2}) ds = (\frac{2c^2}{\alpha} + \alpha)\int_{S-S}^S f(S-s) ds = (\frac{2c^2}{\alpha} + \alpha)\int_o^s f(u) du.$$

By Gronwall's inequality $f\,(t) = o \le s \le S$.n $Y_s^* = o, \le s \le S.$

The comparison theorem explains naturally the option pricing facts in financial market. It makes market prices greater than contingent claim,Y_s^* is greater at the present time.

3. Main Result

Theorem 3.1: Let $\xi \ge 0$ be a positive square integrable contingent claim. Under the assumption mentioned above, then there exist a unique replication strategy (X,π) of ξ such that

$$dX_t = [Y_t X_t + \pi_t\sigma_t\theta_t]dt + \pi_t\sigma_t dB_t, X_T = \xi. \tag{3.1}$$

Hence X is the fair price of the claim, and $X_t$is given by $X_t = \mathbb{E}(H_T\xi|\,\mathcal{F}_t)$, where H_t is the process defined by the forward $LSDE$,

$$dH_t = -H_t[Y_t dt + \theta_t dW_t]\ , H_0 = 1. \tag{3.2}$$

Proof: By assumption, we set that market is arbitrage free and complete on $[0,T]$.Then it implies ξ is replicable, which follows that there exist an initial X_0 and an admissible portfolio $\pi(\cdot)\epsilon\ \mathbb{L}_t^2(\mathbb{R})$such that the corresponding $X(\cdot)$ satisfies $X(T) = \xi.$

Since Y_t and θ_t are bounded by proposition (2.1) it follow there exist an unique solution pair $(X,\sigma,\pi)\ \epsilon K_t^2(\mathbb{R}) \times H_t^2(\mathbb{R}^n)$, satisfying

$$dX_t = [r_t X_t + \pi_t\theta_t\pi_t]dt + \pi_t\sigma_t dB_t, X_t = \xi, \tag{3.3}$$

such that$\int_0^T |\,\pi_t\sigma_t|^2\, dt < +\infty.$

Therefore (X,π) is the unique replication strategy of ξ, $X_t = \mathbb{E}(H_{T\xi}|\, \mathfrak{f}_t)$, with
$dH_t = -H_t[X_t dt + \theta_t dW_t]H_0 = 1$.
Theorem 3.2:Let the general setting of the wealth equation

$$dX_t = -f\left(t\, X, \pi_t, \sigma_t\right) + \pi_t\sigma_t dB_t \tag{3.4}$$

where f is a real process defined on $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n$ satisfying the Lipschitz condition. Suppose$X$ is the wealth process associated with an admissible strategy which finances the contingent claim ξ and (X,σ,π) is the square integrable solution of the BSDE if

$dX_{t=} - f\left(t, X_t, \pi_t, \theta_t\right) + \pi_t\sigma_t dB_t,\ X_{t=\xi}$.

The proprieties hold.

1. The prices X is increasing with respect to the contingent claim ξ
2. If the $\xi \geq 0$ and $f(t,0,0) \geq 0$ then the prices is nonnegative

Proof : Let (X,σ,π) and $\epsilon\, K^2(\mathbb{R}) \times H^2(\mathbb{R})^n$ behe unique adapted solution of

$$dX_t = -f(t, X_t\pi_t\sigma_t) + (X,^- \sigma, \quad \pi)\pi_t\sigma_t dB_t, X_t = \mathbb{E}$$

$d\overline{X}_t = -f\left(t, \overline{X}_t\bar{\bar{\pi}}_t\bar{\bar{\theta}}_t\right) + \overline{\pi_t\overline{\sigma}_t} dB_t, X_t = \mathbb{E}$.

If $\leq \bar{\xi}$, together with $f(t, \overline{X_t}, \pi_t, \sigma_t) = f(t, \overline{X_t}\overline{\pi_t\sigma_t})$ then by comparison theorem, that is $X_t \leq \bar{X}_t$.Hence the price X is the increasing with respect to the contingent claim ξ.

The second part

Let (X,σ,π) and $(0,0)\ \epsilon\ K^2(\mathbb{R}) \times H^2(\mathbb{R}^n)$ be the unique adapted solution
$dX = -f(t, X_t\pi_t\sigma_t) + \pi_t\sigma_t dB_t,\ X_t = \xi, d\bar{X}_t = 0, \bar{X}_T = 0$.
Since$\xi \geq 0$ and $\int(t,0,0) \geq 0$ by comparison theorem m$X_t \geq \bar{X}_t = 0$, hence the price is nonnegative.

Remark: The return on such investment is riskless, as such an asset has deterministic return rata, invertors know how much money he is going to end up with when inverting in the market, but not in advance.

(3.1) CONTINGENT CLAIMS OPTION PRICING IN THE CONSTRAINED CASE

Nonlinear backward equation for the pricing of contingent claims with constrains on the wealth or portfolio process, can be apply to two simple examples in finance through comparison theorem. Suppose in a market there two asset, where the first one is riskless while the other is risky.

(3.1.1) REPLICATING CLAIMS WITH DIFFERENCE RISK PREMIUM FOR LONG AND SHORT POSITION

Suppose in the market there is trading with different risk premium for long and short position. Let$\theta^1 - \theta^2$ be the difference in excess return between long and short position in the stock market, where θ^1 and θ^2 are predictable and bounded contingent claim ξ.There exist a unique square integrable replication strategy $(X,\sigma^*\pi_t)$ which satisfies
$dX_s[\hat{\pi}_s\sigma_s\theta_s^1 + (\hat{\pi}_s)\sigma_s(\theta_s^1 - \theta_s^2)]ds + \hat{\pi}_s\sigma_s dB_s,\ X_s = \xi,$
where X_t is the fair price of the contingent claim ξ at the time t $(X\ \sigma\,\pi)$ *be the solution of the LBSDE*
$dX_s[\hat{\pi}_s\sigma_s\theta_s^1 + (\hat{\pi}_s)\sigma_s(\theta_s^1 - \theta_s^2)]ds + \hat{\pi}_s\sigma_s dB_s,\ X_s = \xi.$
To find a sufficient condition that $X_s = X$
Proposition 3.3.2.That the coefficient $Y_s, \theta_s^1, \theta_s^2, \sigma_s^1$ are deterministic function of t and that $\epsilon\, D^2$. If $(\sigma)^{-1}D_k\,\xi \leq$ 0the price for ξ is $X = \bar{X}$
Proof: Let$(X^R,\pi,R)\ \epsilon\ L^2(0,T,(D^2) \times (D^2)$, and for $1 \leq i \leq m$, aversion of

$$\{(D_v^i X_s^R D_u^i \pi_s^R), 0 \leq u \leq t \leq T\}$$

$dD_u^i X_s^R = -[r_s D_u^i X_s^R + D_u^i\pi_s^R\sigma_s\theta_s^i + \sigma_s(\theta_s^i - \theta_s^2)]dt + D_u\pi_s^R{}_{\sigma_s} dB_s, dD_u^i X_s^R = D_u^i\xi,$
$Y_s^u = (\sigma_u^x)^{-1}D_u X_s^R$ and $Z_s^u = (D_u\pi_s^R)(\sigma_u)^{-1}$, for$0 \leq u \leq s \leq S$. $(Y_s^u, Z_s^u, u \leq s \leq S)$ is the solution of the BSDE
$dY_s^u - [-r_S Y_s^u - Z_s^u\sigma^s\theta_s^1 - Z_s^u\sigma_s(\theta_s^1 - \theta_s^2)]ds + Z_s^u\sigma_s dB_s, Y_s^u = (\sigma_m^*)^{-1}D_m^i\xi.$
Appling the comparison theorem to (Y^u, Z^u) and b$(t,0,0)$ so that if $(\sigma_u^*)^{-1}D_v^1\xi \leq (1)$
then $Y_u^u \leq 0\ (\sigma_u^*)^{-1}D_u X_u^R \leq 0$. And if $\sigma_k^t\pi_k = D_k X_k^R$ then we get $\pi_k \leq 0$.
Therefore the price is non-negative for ξ is $X = X^R$. The predictable price is positive and greater than the positive contingent claim ξ the price fluctuates as quickly as possible, respect to stock market price of

contingent claim. There is difference between the current price of contingent claim and price in the previous price. Market prices are not affected by purchases of the small investment.

3.1.2. Replicating Claims with High Interest Rate for Borrowing

In the market, where an investor is allowed to borrow money in the bank at time t at an interest rate$R_s > r_s$ where r_s is the bond rate and R_s is a predictable and bounded process.Hence the amount borrowed at time t is equal to$(Y_s - \pi_s)^-$ the given a square integrable contingent claim ξ there exists a unique square integrable replication strategy (Y, σ, π_s) which satisfies wealth portfolio.

$dY_t = [r_s Y_s + \pi_s \sigma_s \theta_s - (R_s - r_s)(Y_s - \pi_s)^-]ds + \pi_s \sigma_s dB_s \text{ , } Y_{s\,=\xi}.$

$Y_s$is the fair price and upper price of the contingent claim ξ at time t and (Y^R, σ^R, π^R) be the solution of the LBSBE;

$dY_s^R = [r_s Y_s^R + \pi_s^R \sigma_s \theta_s -](R_s - r_s)(Y_s^R - \pi_s^R)ds + \pi_s \sigma_s dB_s, Y_s^R = \xi,$

$dY_s^R = r_s Y_s^R dB_s + \pi_s \sigma_s \theta_s dB_s = (R_s - r_s)(Y_s^R - \pi_s^R)dB_s + \pi_s \sigma_s dB_s$

Proposition 3.1: That the coefficient $r_s, R_s \theta_s$ and σ_s are deterministic function of s and suppose that $\xi \in D^2$ and if $(\sigma_k^*)^- D_k \xi \geq \xi$ then the prices for ξ is Y=Y^R.

Proof: In order to show, it is sufficient to prove that$(\sigma_k^*)^- D_k Y^R$ and $(\sigma_k)^* D_k \xi \geq \xi$.

Recall that $(Y^R, \pi^R)$is solution of the BSDE

$-dY_s^R = [-R_s Y_s^R - (\pi_s^R)^* (\sigma_s \theta_s + (r_s - R_s)]ds - (\pi^R)^* \sigma_{s\, dB} Y_s^R \quad = \xi$.

Therefore $(Y^R, \pi^R) \epsilon\, C^2(0,T)(D^2 \times D^2)$ for $i \leq 1 \leq n$ a version of $\left(D_k^i \times X_s^R, D_K \pi_s^R\right)$, $0 \leq k \leq s \leq S$ is

$-dD_k^i Y_s^z = -R_s D_s^i Y_s^R - \left(D_k^i \pi_s^R\right)^* - (\sigma_s \theta_s + (r_s - R_s), ds - \left(D_k^i \pi_s^R\right)^* \sigma_s dB, D_s^i Y_s^R = D_k^i \xi$

and$Z_s^R = (D_k \pi_s^R)(\sigma_k)^-for0 \leq k \leq s \leq S$.

That is (Y_S^R, Z_S^R) $k \leq s \leq S$ is the solution of BSDE

$-dY_s^R = -R_s Y_s^R - (Z_s^R)(\sigma_s \theta_s + (r_s - R_s)ds - (Z_s^R)\sigma_s dB, \; Y_s^R = \sigma_k^{*-} D_k \xi, Y_s = Y_s^R.$

Therefore

$$
\begin{aligned}
&(Y_s - \pi_s) r_s ds - (Y_s - r_s)(Y_s - \pi_s)\sigma_s dB \\
&\qquad = \pi_s^2 (\sigma_s dB_s + \mu_s\, ds) + (Y_{s\,-}\pi_s) r_t Y_t\, dt - (R_s - r_s)(Y_s - \pi_s)\, dB \\
&\qquad = r_s Y_s\, ds + \pi_s \sigma_s \theta_s dt + \pi_s \sigma_s dB - (R_s - r_s)(Y_s - \pi_s) dB,
\end{aligned}
$$

which is the extra cost when borrowing.

The interest in borrowing is equal to that of amount borrowing in the bank. The contingent claim is evaluated under a current rate R and a risk premium equal to $\theta_s + (r_s - R_s)ds$. The risk premium is higher than the primitive oneθ_s .

Remark: It is not reasonable to borrow money in the bank to invest money in the bond at the same time. Therefore we restrict ourselves to policies for which the amount borrowed at time is equal to the strategy of wealth, portfolio. The gain in interest when an investor lends money by buying a riskless ass

4. Hedge Portfolio Constructed by Longing a Contingent Claim and Shorting the Unit

In a bid to making the portfolio value riskless where r is a riskless rate we have

$-dY_s = -r_s Y_s ds - Z_s \lambda ds - Z_s ds$ with terminal condition$Y_S = H^R$, where H^R is the discounted contingent claim pay off, Y_t will represent our portfolio as the$-dY_s = +(Y_s - \sum_{i=1}^d \sigma_s^i$.

Constant interest rate $r_t = r$, the risk premium$\lambda_t = \lambda$ and the Volatility$\sigma_t = \sigma$.

Therefore if $\exp\left(-r - \frac{1}{2}\lambda^2\right) dt - \lambda dB_t$, then $Y_t = \in \frac{1}{T_t}\left[\frac{T_t}{T_t} \mid \mathfrak{f}_t\right] = \mathbb{E}\mid \frac{T_t}{T_t} H \mid \mathfrak{f}_t$

$$
\begin{aligned}
&= \mid \exp\left(-r - \frac{1}{2}\lambda^2\right)(T - t) - \lambda(W_t - W_t) H \mid \mathfrak{f}_t \mid \\
&= \mathbb{E}\left(-\int_0^T \lambda dB\right) = \exp\left(\frac{1}{2}\int_0^T \lambda^2 ds - \int_0^T \lambda dB\right).
\end{aligned}
$$

This is the usual risk neutral measure that is density processD_t with

$D_t = \mathbb{E}\mid \mathfrak{f}_t \mid = \mathbb{E}\left(-\int_0^T \lambda dB\right) = \exp\left(-\lambda W_t - \frac{1}{2}\lambda_t^2\right)$,

$Y_t = \xi\,[\exp[-r(T - t)H \mid \mathfrak{f}_t]$.

Remark, this is to normalization of the risk asset to unity, the value is discounted payoff of the contingent claim under risk neutral measure.

4.1. Option Pricing In the Constrained Cases

BSDEs are useful tools in pricing theory using viscosity solution, since they give a generalization of Black Sholes formula, in the cases where the price of a contingent claim which only depend on the prices of the basic securities has the same property [6]. Also the hedging portfolio depends only on these prices. Let consider a financial market with coefficients which only depend on time and on the premium of stock price process $(0.t) \in [O,T] \mathrm{X}\mathbb{R}^d$, the price of the basic securities satisfy the following equation on [0, t]

$$dP_t^i = \boldsymbol{\Upsilon}(t, P_t)\, P_t^o\, dt, \tag{4.1}$$

$$dP_t^i = [\varphi^i(SP_t)dt + \textstyle\sum_{j=i}^n \sigma_j\, (SP_t) dw_t^i. \tag{4.2}$$

Let($P_{t,}^{t,x}$ t ≤ X≤ T) $and\ P_t^{tx} = (P_t^o, P_t^i \cdots\cdots P_t^n)$be the stock price processes with initial condition given by $P_t^{tx} = \chi$. Therefore the general setting of the wealth equation is

$$-dX_t = b\,(t, P_{t,} X_t\ \sigma\,(t, P_t)\,\pi_s)\,dt - \pi_t \sigma\,(SP_t)\,dw_t. \tag{4.3}$$

Here, b is an $\mathbb{R}-$valued continuous function on [o,t] x $\mathbb{R}^n$x $\mathbb{R}$ x $\mathbb{R}$ that is Lipschits with respect to (X, π) uniformly in T.If the super prices of contingent is higher than initial valued, then

$$\frac{\partial y}{\partial x}(t,x) - \mathrm{L}\varphi(\mathrm{t},\mathrm{x}) - fi(t,x,y(t,x), \nabla\varphi\sigma(t,x) \geq o. \tag{4.4}$$

And if decreased in stock price then contingent $\xi > o$ such that

$$\frac{\partial y}{\partial x}(t,x) - \mathrm{L}\varphi(\mathrm{t},\mathrm{x}) - fi(t,x,y(t,x), \nabla\varphi\sigma(t,x) \leq o, \tag{4.5}$$

where θ(t,) is the risk premium vector. θ(t,X) $= \theta^{-1}$(t,X) ((t,X) − r (t,X) is the contingent claim with $\xi = \Phi\,(P_T^{t,x})$.Here $\Phi = \mathbb{R}^{n+1} \to \mathbb{R}$ is Lipschitz. There exist a unique square−integrable hedging strategy ($X^{t,x}$, $\pi^{t,x}$) $\in H_T^2$ x H against ξ such that

$$-dX_t^{t,x} = b\,(t, P_t^{t,x}, X_t^{t,x}, \sigma\,(t, P_t^{t,x})\ \pi_t^{t,x})\,dt - (\pi_t^{t,x})\,\sigma\,(t, P_t^{t,x})\,dw_t\ ,\ X_T^{t,x} = \Phi\,(P_T^{t,x}). \tag{4.6}$$

And $X_S^{t,x}$ is the price of the contingent claim $\Phi\,(P_T^{t,x})$ at time s. the value at time of the contingent claim ξ is;

$$X_s^{t,x} = u\,((s, P_s^{t,x}), \tag{4.7}$$

where $u\,(t,x) = X_T^{t,x}$ is the uniqueviscosity solution of the nonlinear parabolic PDE;

$$\left(\frac{\partial\varphi}{\partial t}\right)(t,x) = \frac{1}{2}\,Tr\,(\sigma\sigma D^2\,U) + (D\,\varphi\,b)\,(t,x)$$

$$= \sum \mathrm{b_{ij}}(t,x)d_{ij} + \sum aij\,(t,x)d_{ij}(t,x)$$

$$= \sum \varphi_\mathrm{j}\,(t,x)x_j \frac{\partial u\,(t,x)}{\partial t} + r\,(t,x)\,x_o \frac{\partial u\,(t,x)}{\partial u_o},$$

$$u\,(T,x) = \Phi\,(x),$$

where $aij\,(t,x) = \frac{1}{2}[\sigma\,\sigma]ij\,(t_{o,} X_O)$ and $x\,\frac{\partial u\,(t,x)}{\partial x} = (xi\,\frac{\partial u\,(t,x)}{\partial x})$. Therefore the portfolio process of the hedging strategy is then

$$\pi_s^i = P_t^i \frac{\partial u\,(t,x)}{\partial x_i}\ (s, P_S)\ \ t \leq s \leq T.$$

5. Conclusion

We have taken advantage of the idea in literature all way through in this work. We clearly illustrated the importance of BSDEs and their application to finance by using the Malliavin and comparison theorems and the theory of BSDE demonstrated to contingent claim in unconstrained constrained cases. Our major target is to have a more realistic and completive market so that the investor can look for a convenient way to raise money and invest in market by bond and riskless asset and also to liquidity on stock return because of unstable pricing and sudden change every day in the market. We evaluated the problem of hedging contingent claim under BSDE of the underlying asset price, by enlarging the market with appropriate futures contract whose payoffs depend on higher order sample mount of the asset price, so that an investor knows how much money he is going to end up with when inverting in the riskless asset.

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