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Overview on Weighted Generalized Entropy, Residual and Past Entropies

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Abstract The concept of entropy was originally introduced in Shannon [27] in the context of communication theory. The main measure of the uncertainty contained in random variable X is the Shannon entropy $H(X)= -E(\ln f_X(x))$. The concept of generalized entropy has been proposed in the literature of information theory. The cumulative entropy is an information measure which is alternative to the differential entropy and is connected with a notion of reliability theory. In this paper, the concept of weighted (generalized) entropy is discussed. The properties of weighted (generalized) entropy, cumulative residual entropy, weighted (generalized) residual entropy, weighted cumulative residual entropy, weighted (generalized) past entropy are also given.

Keywords Generalized entropy, genralized residual entropy, generalized past entropy, weighted cumulative residual entropy, weighted generalized entropy

1. Introduction

We live in an era of extreme uncertainties. The markets are unstable, stock exchange has become more volatile than ever, marketing is experiencing difficulties in persuading new customers to buy more products or services, competition between corporations is fierce and yet in these conditions, managers are called to undertake more risks, in order to produce better results that will increase the trust of the stakeholders and attract new investments [25]. An important measure of uncertainty associated with a random variable X is the notion of entropy, introduced by Shannon [27]. If X is a non-negative random variable having an absolutely continuous distribution function $F_X(x)$ with probability density function $f_X(x)$, then the Shannon's entropy is defined as

$$H(X) = -E\left[\ln f(X)\right] = -\int_0^\infty f_X(x)\ln(f_X(x))dx \quad (1)$$

One of the main drawback of $H(X)$ is that for some probability distribution it may be negative and then it is no longer an uncertainty measure (see Das [6]).This drawback is removed in the generalized entropy. $H(c+X) = H(X)$ for some constant c. This property can be interpreted as the shift-independence of Shannon information. The integrand function on the right-hand-side of (1) depends on x only via $f_X(x)$, thus making H shift-independent. Hence, H stays unchanged if, for instance, X is uniformly distributed in (a,b) or $(a+h, b+h)$, whatever $h \in R$.

However, in certain applied contexts, such as reliability or mathematical neurobiology, it is desirable to deal with shift-dependent information measures. Shannon's entropy gives equal importance or weight to the occurrence of every event. Shannon [27] is considered to be the father of information theory and was the first that incorporated the term information entropy in an information systems for measuring the uncertainty associated with a random variable.

Example 1. Consider a general uniform distribution with the probability density function

$$f_X(x) = \begin{cases} \frac{1}{a}, 0 \le x \le a \\ 0, otherwise \end{cases}$$

Then

$$H(X) = -\int_0^\infty f_X(x) \ln f_X(x) dx = -\int_0^a \frac{1}{a} \ln \frac{1}{a} dx = \frac{1}{a} \ln a \int_0^a dx = \ln a.$$

In the area of information theory as well as engineering sciences the Shannon's entropy and its applications is a very important and well known concept. Information theory includes the study of uncertainty measures and various practical and economical methods of coding information for transmission. It measures the expected uncertainty contained in probability density function about the predictability of an outcome of *X*. Study of duration is a subject of interest in many fields of science such as reliability, survival analysis, economics and business. In reliability theory and survival analysis, the additional lifetime given that the component has survived up to time *t* is called the residual life function of the component. It measures the expected uncertainty contained in probability density function about the predictability of an outcome of *X*. The properties and virtues of *H*(*X*) have been thoroughly investigated by Shannon [27] and Wiener [29]. The basic principle of generalized maximum entropy (GME) is based on Jaynes' Maximum Entropy Principle Jaynes [16] Golan, Judge and Miller generalized this principle for regression framework Golan, Judge & Miller [12]. With this method, Shannon's entropy formula was maximized under model consistency constraints and it was assumed that no prior information about parameters and disturbances would exist. The GME method, requiring less assumptions than the classical methods, has been frequently used for both linear and nonlinear estimation models Golan, Judge & Miller [12] and receiving increasing attention especially in the econometrics and statistics literature. Akdeniz et al, [1] proposed an alternative solution when the data have the problem of multicollinearity. Some constraints were added to the classical generalized maximum entropy approach according to the characteristics of the relationship among independent variables and the results were compared with *OLS*.

The rest of the paper is organized as follows. In Section 2 we provide some basic notions on weighted entropy, complemented by some examples. Section 3 is devoted to study results for the weighted generalized entropy, weighted residual entropy, cumulative residual entropy and weighted past entropy. Analogously, in Section 4 we study the weighed generalized entropy for the discrete random variable *X*. Some conclusions are given in Section 5.

2. Weighted Entropy

The definition and initial results on weighted entropy were introduced in Belis and Guiasu [3] and Guiasu [13], Guiasu [14] has shown that weighted entropy has been used to balance the amount of information and the degree of homogeneity associated to a partition of data in classes. DiCrescenzo and Longobardi [8] have considered a length-biased shift-dependent information measure, related to the differential entropy.

Definition 1. (Differential entropy) The differential entropy *H*(*X*) of a continuous random variable *X* with probability density function (pdf) $f_X(x)$ is defined as

$$H(X) = -\int_S f_X(x) \ln f_X(x) dx$$

where *S* is the support region of the random variable. Weighted entropy, which is a generalization of classical entropy, has been proposed by Belis and Guiasu [3] and is defined in Eq. (2). Other measures of uncertainty as suitable generalizations or modifications of the classical entropy have been proposed in the recent literature such as **the weighted entropy** (Di Crescenzo and Longobardi [8], defined as

$$H^{\omega}(X) = -E(X \ln f(X)) = -\int_0^\infty x f_X(x) \ln f_X(x) dx, \quad (2)$$

The factor x, in the integrand of Equation (2) represents a weight which linearly emphasizes the occurrence of the event $\{X=x\}$. *This is* a "length-biased" shift-dependent information measure assigning greater importance to larger values of *X*. When the weight function depends on the length of the component, the resulting distribution is called length-biased weighted function [21].

Several applications with weighted entropy were performed in the middle eighties. Batty [2] used weighted entropy to discuss the spatial pattern of aggregation in cities; Nawrocki and Harding [24] used state-value weighted entropy as a measure of investment risk. Casquilho [5] discussed a framework combining traditional expected utility and weighted entropy.

Example 2. a) *X* is exponential distributed with parameter $\lambda > 0$.

Its probability density function is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0 \\ 0, otherwise \end{cases}$$

A standard agreement 0 = 0 · log 0 = 0 · log ∞ is adopted. Then

$$H^{\omega}(X) = -\int_0^{\infty} x f_X(x) \ln(f_X(x)dx = -\int_0^{\infty} x\lambda e^{-\lambda x} \ln(\lambda e^{-\lambda x})dx$$

$$= -\ln\lambda \int_0^{\infty} \lambda x e^{-\lambda x} dx + \int_0^{\infty} (\lambda x)^2 e^{-\lambda x} dx = \frac{2 - \ln\lambda}{\lambda}$$

b) If *X* uniformly distributed over $[a,b]$.

$$f_X(x) = \begin{cases} \frac{1}{b-a}, a \le x \le b \\ 0, otherwise \end{cases}$$

Then

$$H^{\omega}(X) = -\int_a^b x \frac{1}{b-a} \ln(\frac{1}{b-a})dx = \frac{\ln(b-a)}{b-a}\int_a^b x dx = \frac{b^2 - a^2}{2}\frac{\ln(b-a)}{b-a}$$

$$= \frac{a+b}{2}\ln(b-a)$$

where $E(X) = \frac{a+b}{2}$

and

$$H(X) = -\int_0^{\infty} f_X(x)\ln(f_X(x))dx = -E[\ln f_X(x)] = -\int_a^b \frac{1}{b-a}\ln(\frac{1}{b-a})dx = \ln(b-a).$$

It is interesting to note that in this case the weighted entropy can be expressed as the product of

$$H^{\omega}(X) = E(X)H(X).$$

Remark 2.1 Notice that in general H^{ω} can be either larger or smaller than *H*. For instance, if *X* is uniformly distributed over $[a,b]$, then it follows $H^{\omega} = \frac{a+b}{2}\ln(b-a) = E(X)\ln(b-a) > H = \ln(b-a)$ when E(*X*) > 1, and H^{ω} <*H* when E(*X*) <1.

Example 3 Suppose *X* and *Y* denote random variables with density functions

$$f_X(x) = \frac{1}{8}x, 0 < x < 4$$

$$g_Y(y) = \frac{1}{8}(4 - y), 0 < y < 4$$

respectively. By simple calculations, we have

$$H(X) = -\int_0^4 f_X(x)\ln f_X(x)dx = -\int_0^4 \frac{x}{8}\ln\frac{x}{8}dx = \ln 2 + \frac{1}{2}$$

$$H(Y) = -\int_0^4 g_Y(y)\ln g_Y(y)dy = -\int_0^4 \frac{1}{8}(4-y)\ln\frac{1}{8}(4-y)dy = \ln 2 + \frac{1}{2}$$

Their Shannon entropies are identical. Therefore, the expected uncertainties for $f_X(x)$ and $g_Y(y)$ on the predictability of the outcomes of the *X* and *Y* are identical. But, we have

$$H^{\omega}(X) = -\int_0^4 xf_X(x)\ln f_X(x)dx = -\int_0^4 x\frac{1}{8}x\ln\frac{1}{8}xdx = \frac{8}{3}\ln 2 + \frac{8}{9}$$

$$H^{\omega}(Y) = -\int_0^4 yg_Y(y)\ln g_Y(y)dy = -\int_0^4 y\frac{1}{8}(4-y)\ln\frac{1}{8}(4-y)dy = \frac{4}{3}\ln 2 + \frac{10}{9}$$

then $H^{\omega}(X) > H^{\omega}(Y)$. Hence, even though $H(X) = H(Y)$, the expected weighted uncertainty contain in of the $f_X(x)$ on the predictability of the outcome of *X* is larger than that of $g_Y(y)$ on the predictability of the outcome of *Y*.

3. Weighted (Generalized) Entropy

Definition 2. Survival function: $S(t) = \bar{F}_X(t) = P(X > t) = 1 - F_X(t)$

If *X* is an absolutely continuous non-negative random variable with probability density function $f_X(.)$ and survival function $\bar{F}_X(.)$, then the probability function of weighted random variable X^{ω} associated to the random variable *X* with weight function *w(x)* is defined by

$$f^{\omega}(x) = \frac{w(x)}{E(w(X))} f_X(x), 0 \le x < \infty$$

where $w(x)$ is positive for all value of $x \ge 0$ and $0 < E(w(X)) < \infty$.The random variable X^{ω} arises in the study of lifetime analysis. On particular choices of weight function *w(x)* we have different weighted models. For example, when *w(x)=x*, resulting distribution is called length biased distribution and the associated probability density function of length biased random variable X^{ω} is defined as

$$f^{\omega}(x) = \frac{x}{E(X)} f_X(x) \tag{3}$$

and the survival function is

$$\bar{F}^{\omega}(x) = \frac{E(X|X > x)}{E(X)} \bar{F}_X(x) \quad [6, 18] \tag{4}$$

Survival functions are most often used in reliability and related fields. The survival function is the probability that the variate takes a value greater than *t*.

We can see that by replacing the original distribution in (1) by the weighted distribution with respect to that original distribution, the corresponding equation becomes

$$H^{\omega}(X) = -\int_0^{\infty} f^{\omega}(x)\ln f^{\omega}(x)dx \tag{5}$$

On substituting the values of weighted functions in Eq. (3), we get

$$H^{\omega}(X) = -\int_0^{\infty} x f_X(x) \frac{(\ln x + \ln f_X(x))}{E(X)} dx + \int_0^{\infty} \frac{x f_X(x)}{E(X)} \ln E(X) dx$$

$$= -\frac{1}{E(X)} \int_0^{\infty} x f_X(x) \ln x dx - \frac{1}{E(X)} \int_0^{\infty} x f_X(x) \ln f_X(x) dx + \frac{E(X) \ln E(X)}{E(X)}$$

$$= \frac{E(X) \ln E(X)}{E(X)} - \frac{E(X \ln X)}{E(X)} - \frac{1}{E(X)} \int_0^{\infty} x f_X(x) \ln f_X(x) dx$$

$$H^{\omega}(X) = c_1 - c_2 \int_0^{\infty} x f_X(x) \ln f_X(x) dx,$$

where $c_1 = -\frac{E(X \ln X)}{E(X)} + \frac{E(X) \ln E(X)}{E(X)}, c_2 = \frac{1}{E(X)}$. Eqn. (5) is the definition of weighted entropy [3]. Now, the weighted generalized entropy is given by

$$H_1^{\omega^{\beta}}(X) = \frac{1}{\beta - 1} \left[1 - \int_0^{\infty} f^{\omega^{\beta}}(x) dx \right]$$

$$= \frac{1}{\beta - 1} \left[1 - \frac{1}{(E(X))^{\beta}} \int_0^{\infty} x^{\beta} f_X^{\beta}(x) \right] \quad (6)$$

and

$$H_2^{\omega^{\beta}}(X) = \frac{1}{1 - \beta} \left[\ln\left(\frac{1}{(E(X))^{\beta}} \int_0^{\infty} x^{\beta} f_X^{\beta}(x) dx, \right] \quad (7)$$

We note that as $\beta \to 1$ in (6) or (7), they reduce to (5). $H_1^{\omega^{\beta}}(X)$ and $H_2^{\omega^{\beta}}(X)$ are called first kind weighted entropy of order β and second kind weighted entropy of order β respectively.

The following example shows that, although two distributions have same generalized entropies, they have different weighted generalized entropies.

Example 4. Let *X* and *Y* be random variables with density functions

$$f_X(t) = \begin{cases} \frac{2+t}{6}, 0 \le t < 2 \\ 0, otherwise \end{cases}$$

and

$$f_Y(t) = \begin{cases} 1 - (2+t)/6, 0 \le t < 2 \\ 0, otherwise \end{cases}$$

Take $\beta = 2$. Then we have

$$H_1^{\beta}(X) = \frac{1}{\beta - 1} \left[1 - \int_0^{\infty} f_X^{\beta}(x) dx \right] = H_1^2(X) = \frac{1}{2-1} \left[1 - \int_0^2 (\frac{2+t}{6})^2 dt \right] = \frac{13}{27}$$

and

$$H_1^{\beta}(Y) = \frac{1}{\beta - 1} \left[1 - \int_0^{\infty} f_Y^{\beta}(y) dy \right] = H_1^2(Y) = \frac{1}{2-1} \left[1 - \int_0^2 (1 - \frac{2+t}{6})^2 dt \right] = \frac{13}{27}$$

Thus we can see that $H_1^{\beta}(X)=H_1^{\beta}(Y)=\frac{13}{27}$, where $H_1^{\beta}(X)$ and $H_1^{\beta}(Y)$ are the first kind generalized entropies of X and Y. The first kind weighted generalized entropies of the random variables X and Y are given by

$$H_1^{\omega\beta}(X)=\frac{1}{\beta-1}\left[1-\int_0^{\infty} f^{\omega\beta}(x)dx\right]=\frac{1}{\beta-1}\left[1-\frac{1}{(E(X))^{\beta}}\int_0^{\infty} x^{\beta} f_x^{\beta}(x)dx\right]$$

$$=\frac{1}{2-1}\left[1-\frac{1}{(E(X))^2}\int_0^2 x^2(\frac{2+x}{6})^2 dx\right]=\frac{256}{1000}$$

and

$$H_1^{\omega\beta}(Y)=\frac{1}{\beta-1}\left[1-\frac{1}{(E(Y))^{\beta}}\int_0^{\infty} y^{\beta} f_Y^{\beta}(y)dy\right]=$$

$$=\frac{1}{2-1}\left[1-\frac{1}{(E(Y))^2}\int_0^2 y^2(1-\frac{2+y}{6})^2 dy\right]=\frac{31}{160}$$

where $E(X)=\int_0^2 xf_X(x)dx=\int_0^2 x\frac{2+x}{6}dx=\frac{10}{9}$ and $E(Y)=\int_0^2(1-\frac{2+y}{6})dy=\frac{8}{9}$,

respectively. Therefore, $H_1^{\omega\beta}(X)=\frac{256}{1000}\neq H_1^{\omega\beta}(Y)=\frac{31}{160}$.

Again, the second kind generalized entropies of the random variables X and Y are given by

$$H_2^{\beta}(X)=\frac{1}{1-\beta}\ln\int_0^{\infty} f_X^{\beta}(x)dx=H_2^2(X)=\frac{1}{1-2}\ln\int_0^2(\frac{2+t}{6})^2 dt=\ln\frac{27}{14}$$

$$H_2^{\beta}(Y)=\frac{1}{1-\beta}\ln\int_0^{\infty}\left[1-\frac{2+t}{6}\right]^2 dt=-\ln\int_0^2(\frac{4-t}{6})^2 dt=\ln\frac{27}{14}.$$

Then, we have $H_2^{\beta}(X)=H_2^{\beta}(Y)=\ln\frac{27}{14}$ .But, second kind weighted generalized entropies of order β are

$$H_2^{\omega\beta}(X)=\frac{1}{1-\beta}\left[\ln(\frac{1}{(E(X))^{\beta}}\int_0^{\infty} x^{\beta} f_X^{\beta}(x)dx)\right]$$

$$=\frac{1}{(1-2)}\left[\ln\{\ \frac{1}{(10/9)^2}\int_0^2 x^2\frac{1}{36}(2+x)^2 dx\ \}\right]=\ln\frac{125}{93}$$

and

$$H_2^{\omega\beta}(Y)=\frac{1}{1-\beta}\left[\ln(\frac{1}{(E(Y))^{\beta}}\int_0^{\infty} y^{\beta} f_Y^{\beta}(y)dy\right]$$

$$=\frac{1}{(1-2)}\left[\ln\left\{\frac{1}{(8/9)^2}\int_0^2 y^2\frac{1}{36}(4-y)^2 dy\right]=\ln\frac{5}{3}$$

$H_2^{\omega\beta}(X)=\ln\frac{125}{93}$ and $H_2^{\omega\beta}(Y)=\ln\frac{5}{3}$ are not equal. Hence, even though $H_1^{\beta}(X)=H_1^{\beta}(Y)=\frac{13}{27}$ and $H_2^{\beta}(X)=H_2^{\beta}(Y)=\ln\frac{27}{14}$, the weighted generalized entropy about the predictability of X by the density function $f_X(t)$ is smaller than the predictability of Y by the density function $f_Y(t)$.

3.1. Weighted Residual Entropy

In recent years, the role of Shannon entropy as a measure of uncertainty in residual lifetime distributions has been studied by many researchers [4, 10-11].

When a unit studied that survived up to an age *t*, the Shannon's entropy is not suitable for measuring the uncertainty. So the notion of residual and past uncertainty has been introduced.

Let X be an absolutely continuous nonnegative random variable having distribution function $F_X(x)=P(X\le x)$ and the survival function $\bar{F}_X(x)=P(X>x)=1-P(X\le x)$.In reliability theory, X represents the random lifetime of an item or system with survival function $\bar{F}_X(.)$.

$\lambda(x)=\dfrac{f_X(x)}{\bar{F}_X(x)}$: the hazard function, or failure rate, of X;

$\tau(x)=\dfrac{f_X(x)}{F_X(x)}$:the reversed hazard rate function of X;

The residual lifetime $X_t=[X-t|X>t]$, $t>0$, describes the time length between the failure time X and the inspection time *t*, given that time *t* the system is still active [17].

Suppose X denotes the lifetime of a component/system or of a living organism and $f(t)=F'(t)$ denotes the lifetime density function [22]. If a component is known to have survived to age *t* then Shannon entropy is no longer useful to measure the uncertainty of remaining lifetime of the component.

Ebrahimi [10] defined the entropy for residual lifetime $X_t=(X-t\mid X>t)$ as a dynamic form of uncertainty called the ***residual entropy*** at time *t* and defined as

$$H(X;t)=H(f,t)=-\int_t^{\infty}\frac{f_X(x)}{\bar{F}_X(t)}\ln(\frac{f_X(x)}{\bar{F}_X(t)})dx \tag{8}$$

$$=\ln\bar{F}_X(t)-\frac{1}{\bar{F}_X(t)}\int_t^{\infty}f_X(x)\ln f_X(x)dx$$

$$=1-\frac{1}{\bar{F}_X(t)}\int_t^{\infty}f_X(x)\ln(\lambda_X(x))dx$$

$$=1-\frac{1}{\bar{F}_X(t)}\int_t^{\infty}f_X(x)\{\ln f_X(x)-\ln\bar{F}_X(x)\}dx$$

$$=1-\frac{1}{\bar{F}_X(t)}\int_t^{\infty}f_X(x)\ln f_X(x)dxdx+\frac{1}{\bar{F}_X(t)}\int_t^{\infty}f_X(x)\ln\bar{F}_X(x)dx$$

$$=1-\frac{1}{\bar{F}_X(t)}\int_t^{\infty}f_X(x)\ln f_X(x)dx+A$$

where

$$A=\frac{1}{\bar{F}_X(t)}\int_t^{\infty}f_X(x)\ln\bar{F}_X(x)dx=\frac{-1}{\bar{F}_X(t)}\int_{\bar{F}_X(t)}^{0}\ln w\,dw=\ln\bar{F}_X(t)-1,$$

$w=\bar{F}_X(x)=1-F_X(x)$, $F_X(\infty)=1$ and $\lambda_X(x)=\dfrac{f_X(x)}{\bar{F}_X(x)}$ is the hazard function, or failure rate, of X; $f_X(x)$ is the probability density function; $\bar{F}_X(t)$ be the survival function of the random variable X. Ebrahimi

[10] showed that $H(X,t)$ uniquely determines the distribution function $F_X(t)$.Obviously *H(X;0)=H(X).* It is well known from (8) that units which exhibitless uncertainty in life times are more reliable and hence measure (8) has much relevance in characterizing, ordering and classifying life distributions according to its behavior.
Nanda and Paul [22] have introduced **generalized residual entropy** and they redefined (9) and (10) for a unit surviving up to age *t* as

$$H_1^{\beta}X;t) = \frac{1}{\beta-1}\left[1-\int_t^{\infty}(\frac{f_X(x)}{\bar{F}_X(x)})^{\beta}dx\right] \tag{9}$$

and

$$H_2^{\beta}(X;t) = \frac{1}{1-\beta}\ln\int_t^{\infty}(\frac{f_X(x)}{\bar{F}_X(t)})^{\beta}dx \tag{10}$$

respectively. As $\beta \to 1$ in (9) or in (10), then they tends to (8).
Di Crescenzo and Longobardi [8] have introduced the concept of **weighted residual entropy** at time *t* of a random lifetime *X* can be defined as

$$H^{\omega}(X,t) = -\int_t^{\infty}\frac{f^{\omega}(x)}{\bar{F}^{\omega}(t)}\ln\frac{f^{\omega}(x)}{\bar{F}^{\omega}(t)}dx \tag{11}$$

We note that H(*X*) is the differential entropy of the residual lifetime of *X* at time *t* , *i.e.*, $\left[X \middle| X > t\right]$. We now make use of (2) to define weighted entropy for residual lifetime that is the weighted version of entropy (8).

$$H^{\omega}(X,t) = -\int_t^{\infty}x\frac{f_X(x)}{\bar{F}_X(t)}\ln\frac{f_X(x)}{\bar{F}_X(t)}dx\,. \tag{12}$$

Example 5. For an exponential distribution with parameter λ > 0, the weighted residual entropy is given by

$$H^{\omega}(X,t) = -\int_t^{\infty}x\frac{f(x)}{\bar{F}(t)}\ln\frac{f(x)}{\bar{F}(t)}dx = -\int_t^{\infty}x\frac{\lambda e^{-\lambda x}}{e^{-\lambda t}}\ln\frac{\lambda e^{-\lambda x}}{e^{-\lambda t}}dx$$

$$= -e^{\lambda t}\int_t^{\infty}\lambda x e^{-\lambda x}(\ln\lambda - \lambda x)dx - \lambda t e^{\lambda t}\int_t^{\infty}\lambda x e^{-\lambda x}dx$$

$$= -e^{\lambda t}(\lambda t + \ln\lambda)\int_t^{\infty}\lambda x e^{-\lambda x}dx + e^{\lambda t}\int_t^{\infty}(\lambda x)^2 e^{-\lambda x}dx\,,$$

$$= -e^{\lambda t}(\lambda t + \ln\lambda)A^* + e^{\lambda t}B^*$$

Since, we have

$$A^* =: \int_t^{\infty}\lambda x e^{-\lambda x}dx = (t+\frac{1}{\lambda})e^{-\lambda t}$$

and

$$B^* =: \int_t^{\infty}(\lambda x)^2 e^{-\lambda x}dx = \lambda t^2 + 2t + \frac{2}{\lambda}$$

then we obtain

$$H^{\omega}(X,t) = -e^{\lambda t}(\lambda t + \ln\lambda)(t+\frac{1}{\lambda})e^{-\lambda t} + e^{\lambda t}(\lambda t^2 + 2t + \frac{2}{\lambda}) = t + \frac{2}{\lambda} - (t+\frac{1}{\lambda})\ln\lambda$$

3.2. Cumulative Residual Entropy: A new measure of information

In this section, we give a brief review and to provide some results, including simple examples of applications to related notions of information theory.

The Shannon entropy has certain disadvantages. For example,it may take any value on the extended real line, it requires the knowledge of density function for non-discrete random variables, the discrete Shannon entropy dose not converge to its continuous analogous, and in order to estimate the Shannon entropy for a continuous density, one has to obtain the density estimation, which is not a trivial task.Recently, Rao et al. [26] introduced an alternative measure of uncertainty called cumulative residual entropy (CRE) which is based on the survival (reliability) function $\bar{F}_X(x) = 1 - F_X(x)$ instead of the pdf$f_X(x)$used in the classical Shannon's entropy (1) [28].

As an alternative measure of uncertainty, Rao et al. [26] proposed the cumulative residual entropy (CRE) of X defined by

$$\varepsilon(X) = -\int_0^\infty \bar{F}_X(x)\ln \bar{F}_X(x)dx \qquad (13)$$

where $\bar{F}_X(x) = 1 - F_X(x)$ is survival function. ε (X) measures the uncertainty contained in the survival function of X. The basic idea in their definition was to replace the density function by the survival function in Shannon's definition. CRE is more general than the Shannon entropy and possesses more general mathematical properties than the Shannon entropy. This measure is always non-negative and its definition is valid for both continuous and discrete cases.

Example 6. (CRE of the Uniform distribution) Consider a general uniform distribution with the probability density function

$$f_X(x) = \begin{cases} \frac{1}{a}, 0 \le x \le a \\ 0, otherwise \end{cases}$$

Then its CRE is computed as follows

$$\varepsilon(X) = -\int_0^a \bar{F}_X(x)\ln\bar{F}_X(x)dx = -\int_0^a (1-\frac{x}{a})\ln(1-\frac{x}{a})dx = \frac{1}{4}a.$$

where $F(x) = P(X \le x) = \int_0^x \frac{1}{a}dt = \frac{x}{a}$ and $\bar{F}_X(x) = 1 - F_X(x) = 1 - \frac{x}{a}$.

Example 7. (CRE of the exponential distribution) The exponential distribution with mean $\frac{1}{\lambda}$ has the probability density function $f_X(x) = \lambda e^{-\lambda x}, x > 0$

Correspondingly, the CRE of the exponential distribution is

$$\varepsilon(X) = -\int_0^\infty \bar{F}(x)\ln \bar{F}(x)dx = -\int_0^\infty e^{-\lambda x}\ln e^{-\lambda x}dx = \int_0^\infty \lambda x e^{-\lambda x}dx = \frac{1}{\lambda}$$

where

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}dt = 1 - e^{-\lambda x}, \bar{F}_X(x) = 1 - F_X(x) = e^{-\lambda x}.$$

CRE has many interesting applications in different branches of sciences such as reliability theory, survival analysis, computer vision, image processing and etc [30].

Misagh et al. [20] and Mirali et al, [19] defined the notions of weighted cumulative residual entropy (WCRE) and weighted cumulative entropy (WCE).

Definition 3. (**Weighted cumulative residual entropy)** Let X be nonnegative continuous random variable having survival function $\bar{F}_X(t)$. We define the weighted cumulative residual entropy (WCRE) of X by

$$\varepsilon^\omega(X) = -\int_0^\infty x\bar{F}_X(x)\ln \bar{F}_X(x)dx \qquad (14)$$

Now we evaluate WCRE of uniform distribution.

Example 8. Let X be uniformly distributed on [0, a], a > 0, then

$$\varepsilon^{\omega}(X)=-\int_0^a x(1-\frac{x}{a})\ln(1-\frac{x}{a})dx=\frac{(a-0)(5a+4.0)}{4.9}=\frac{a}{4}.\frac{5a}{9}=\varepsilon(X).\frac{5a}{9}.$$

If $\frac{5a}{9}<1(>1)$, then $\varepsilon^{\omega}(X)<(>\varepsilon(X))$ and if $5a=9$ then $\varepsilon^{\omega}(X)=\varepsilon(X)$.

3.3. (Weighted) Past Entropy

In some practical situations, uncertainty is related to past life time rather than future. As an example, one can be find past uncertainty of a unit that failed at time *t*.

Di Crescenzo and Longobardi [7] have introduced past entropy over (0,*t*). Since it is reasonable top resume that in many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. They have also shown the necessity of past entropy and its relation with the residual entropy. If X denotes the lifetime of an item or of a living organism, then **past entropy** (or uncertainty of lifetime distribution) of an item is defined as

$$\bar{H}(X;t)=-\int_0^t \frac{f_X(x)}{F_X(t)}\ln(\frac{f_X(x)}{F_X(t)})dx \tag{15}$$

Note that (15) can be rewritten as

$$\bar{H}(X,t)=F_X(t)-\frac{1}{F_X(t)}\int_0^t f_X(x)\ln(f_X(x)dx. \tag{16}$$

where $F_X(t)$ be the distribution function of the random variable X.Given that at time t an item has been found to be failing, $\bar{H}(X,t)$ measure the uncertainty about its past life.

Nanda and Paul [23] have studied some properties and applications of past entropy. Gupta and Nanda [15] have defined generalized past entropies by

$$\bar{H}_1^{\beta}(X;t)=\frac{1}{\beta-1}\left[1-\int_0^t(\frac{f_X(x)}{F_X(x)})^{\beta}dx\right] \tag{17}$$

and

$$\bar{H}_2^{\beta}(X;t)=\frac{1}{1-\beta}\ln\int_0^t(\frac{f_X(x)}{F_X(t)})^{\beta}dx \tag{18}$$

respectively.

Motivated by the salient features of (13), Di Crescenzo and Longobardi [9] proposed a dual concept of CRE called **cumulative past entropy** (CPE) defined as

$$\bar{\varepsilon}(X)=-\int_0^{\infty}F_X(x)\ln F_X(x)dx \tag{19}$$

which measures information concerning past lifetime.

Example 9 If X is uniformly distributed in $[0,a]$, then

$$\bar{\varepsilon}(X)=-\int_0^{\infty}F_X(x)\ln F_X(x)dx=\int_0^a\frac{x}{a}\ln\frac{x}{a}dx=\frac{a}{4}=\varepsilon(X)$$

due to Example 6.

Di Crescenzo and Longobardi [8] have defined weighted past entropy.The **weighted past entropy** at time t of a random lifetime X is defined as

$$\bar{H}^{\omega}(X,t) = -\int_0^t \frac{f_X^{\omega}(x)}{F_X^{\omega}(t)} \ln(\frac{f_X^{\omega}(x)}{F_X^{\omega}(t)})dx. \tag{20}$$

We note that H(*X*) is the differential entropy of the past lifetime of X at time t, i.e.,$\left[X \middle| X \le t\right]$. We now make use of (2) to define weighted entropy for past lifetime that is the weighted version of entropy (15).

$$\bar{H}^{\omega}(X,t) = -\int_0^t x\frac{f_X(x)}{F_X(t)} \ln(\frac{f_X(x)}{F_X(t)})dx \tag{21}$$

Example 10. If *X* is uniformly distributed on (0,b). The weighted past entropy is

$$\bar{H}^{\omega}(X,t) = -\int_0^t x\frac{f_X(x)}{F_X(t)} \ln(\frac{f_X(x)}{F_X(t)})dx = -\frac{1}{F_X(t)}\int_0^t x\frac{1}{b}\ln(\frac{\frac{1}{b}}{F_X(t)})dx$$

$$= -\frac{1}{F_X(t)}\int_0^t \frac{1}{b}x\left[-\ln b - \ln F_X(t)\right]dx = \frac{\ln F_X(t) + \ln b}{bF_X(t)}\int_0^t xdx$$

$$= \frac{\ln b + \ln\frac{t}{b}}{b\frac{t}{b}}\frac{t^2}{2} = \frac{1}{2}t\ln t$$

Example11. The weighted past entropy of an exponentially distributed random variable with parameter $\lambda > 0$ is given by

$$\bar{H}^{\omega}(X,t) = -\int_0^t x\frac{f_X(x)}{F_X(t)} \ln(\frac{f_X(x)}{F_X(t)})dx =$$

$$= -\frac{1}{F_X(t)}\int_0^t xf_X(x)\ln f_X(x)dx + \frac{\ln F_X(t)}{F_X(t)}\int_0^t xf_X(x)dx$$

$$= \frac{\ln F_X(t) - \ln\lambda}{F_X(t)}\int_0^t \lambda xe^{-\lambda x}dx + \frac{1}{F_X(t)}\int_0^t (\lambda x)^2 e^{-\lambda x}dx$$

$$= \frac{1}{F_X(t)}\{\ \ln\frac{F_X(t)}{\lambda}\left[\frac{1}{\lambda} - (t+\frac{1}{\lambda})e^{-\lambda t}\right] + \frac{2}{\lambda} - \left[\lambda t^2 + 2(t+\frac{1}{\lambda})\right]e^{-\lambda t}\ \}$$

$$= \frac{1}{1-e^{-\lambda t}}\left\{\frac{2}{\lambda} - \frac{2}{\lambda}e^{-\lambda t} - 2te^{-\lambda t} - \lambda t^2 e^{-\lambda t} + \left[(\frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda t} - te^{-\lambda t}\right]\ln\frac{1-e^{-\lambda t}}{\lambda}\ \right\}$$

4. Conclusions

This paper gives a formula for the differential entropy (Shannon's entropy) as a measure of uncertainty supplied by a probabilistic experiment depending both on the probabilities of events and on qualitative weights of possible. The concept of weighted generalized entropy has been discussed and some new examples are given. In literature of information measures, generalized entropy is a famous concept which always give a nonnegative uncertainty measure. The several results on the first and second kind of generalized entropies have been discussed. But in many survival studies for modeling statistical data information about lifetime is available. Weighted residual entropy, cumulative residual entropy, weighted cumulative residual entropy, and weighted past entropy is also discussed.

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