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Research Article

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Numerical Treatment for Solving Fractional Riccati Differential Equation using VIM – Restrictive Padé

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Abstract The vim- Restrictive Padé is modification of vim-Padé which we make truncation of series given by vim then we find the Padé approximation and Restrictive Padé approximation. the results between vim, vim-Padé, vim- Restrictive Padé are compered and show that vim- Restrictive Padé gives high accurate than the truncation series by vim and also better than vim- Padé.

Keywords vim- Restrictive Padé-Fractional Differential Equation- Riccati Differential Equation

1. Introduction

The Riccati differential equation was introduced by the Italian Nobleman Count Jacopo Francesco Riccati (1676-1754). More than a quarter of a millennium ago, the Riccati equation has been widely studied in the last few decades. Bittanti et al. [1] in 1992 introduced ahistorical survey and solution of both continuous and discrete Riccati equation. Reid [2] presents a brief survey of basic properties of scalar Riccati differential equations, Riccati matrix and described some occurrences of Riccati matrix differential equations in various applications.Since its introduction in control theory in the sixties, the Riccati equation has known an impressive range of applications in both classical and modern science and engineering ,such as random processes and optimal control[3-5], transmission line phenomena , optimization and robust stabilization, stochastic realization, synthesis of linear passive networks, calculus of variations[6-9] to name but a few.

As it is well known, Riccati differential equations concerned with applications in pattern formation in dynamic games, linear systems with Markovian jumps, river flows, econometric models, stochastic control, theory, diffusion problems, and invariant embedding [10–13].

The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional equation reduces to the classical Riccati differential equation. The importance of this equation usually arises in many engineering areas. in the optimal control problems. The feedback gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation which has to be found for the whole time horizon of the control process [14].

The existing literature on fractional differential equations tends to focus on particular values for the order α. The value $\alpha = \frac{1}{2}$is especially popular. This is because in classical fractional calculus, many of the model equations developed used these particular orders of derivatives [15].Many studies have been conducted on solutions of the Riccati differential equations. Some of them, the approximate solution of ordinary Riccati differential equation obtained from homotopy perturbation method (HPM) [16,17], homotopy analysis method (HAM) [18, 19].

He in [20, 21] introduced thevariational iteration method (VIM)], was successfully applied for both ordinary and partial differential equations and other fields. He was starting to apply the variational iteration method to fractional differential equations[22]. In recent years Modification of the homotopy perturbation method[23].Thevariational iteration method [24, 25] to solve quadratic Riccati differential equation of fractional order.and Modification on Decomposition method for solving fractional Riccati by transform the solution to Padé approximation is done in [26],Chebyshev finite difference introduced by Khader in [27].

We consider here the following non-linear fractional order Riccati differential equation

$$D_*^{\alpha} y(t) = A(t) + B(t)y + C(t)y^2 \quad t > 0, \qquad n-1 \le \alpha \le n \tag{1}$$

Subject to the initial conditions

$$y^{(k)}(0) = c_k \qquad k = 0,1,\dots,n-1 \tag{2}$$

whereα is the fractional order derivative and n is an integer. $A(t), B(t), C(t)$are given real functions $c_k \quad k = 0,1,\dots,n-1$ is constant.

2. Fractional Order Calculus

Fractional calculus deals with derivatives and integrals of arbitrary order and conceder a generalization of classical calculus. Fractional calculus deals with derivatives and integrals of arbitrary orderprovides a more powerful tool for modeling the real live phenomena, and this is actually a natural result of the fact that in FC the integer orders are just special cases.

Definition: Let $\alpha \in R^+$. The operator J_a^{α} defined on $L1[a,\ b]$ by

$$J_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{3}$$

for $a \le t \le b$, is called the Riemann-Liouville fractional integral operator of order α

Definition: Let $\alpha \in R^+$ and $n = \lceil \alpha \rceil$. The operator D_a^{α} defined as

$$D_a^{\alpha} f(t) = D^n J_a^{n-\alpha} f(t) \tag{4}$$

$$D_a^{\alpha} f(t) = \begin{cases} D^n \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau & n-1 < \alpha < n \\ \dfrac{d^n}{dt^n} f(t) & \alpha = n \end{cases} \tag{5}$$

for $a \le t \le b$, is called the Riemann-Liouville differential operator of order α.

the Riemann-Liouville differential operator is the left-inverse operator of the Riemann-Liouville fractional integral operator

i.e

$$D_a^{\alpha} J_a^{\alpha} = I$$

by convention

$$D_a^0 f(t) = f(t) \qquad i.e \qquad D_a^0 = I$$

Definition: Let $\alpha \in R^+$ and $n = \lceil \alpha \rceil$. The operator D_{*a}^{α} defined by

$${}_a^c D_x^{\alpha} f(t) = D_{*a}^{\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau & n-1 < \alpha < n \\ \dfrac{d^n}{dt^n} f(t) & \alpha = n \end{cases} \tag{6}$$

for $a \le t \le b$, is called the Caputo differential operator of order α

3. Analysis of the Variational Iteration Method

Consider the differential equation

$$Ly + Ny = g(t) \tag{8}$$

Where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term. Apply the variational iteration method to equation (8) to find the correction functional on the form:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\xi)\big(Ly_n(\xi) + N\tilde{y}_n(\xi) - g(\xi)\big)\, d\xi \tag{9}$$

Taking the fractional differential equation in the form:

$$D_*^{\alpha}\mathrm{y(t)} = \mathrm{A(t)} + \mathrm{B(t)}y + C(\mathrm{t})y^2, \qquad 0 < \alpha \leq 1, \tag{10}$$

with initial condition $y(0) = 0$, where $D^{\alpha} = d^{\alpha}/dt^{\alpha}$. According to the variational iteration method [9], we construct a correction functional for (10) which reads

$$y_{n+1} = y_n + I^{\alpha}\lambda(\xi)\left[\frac{d^{\alpha}y_n}{d\xi^{\alpha}} - \mathrm{A(t)} - \mathrm{B(t)}y_n - C(\mathrm{t})y_n^2\right] \tag{11}$$

To identify the multiplier, we approximately write (11) in the form

$$y_{n+1} = y_n + \int_0^t \lambda(\xi)\left[\frac{d^{\alpha}y_n}{d\xi^{\alpha}} - \mathrm{A(t)} - \mathrm{B(t)}\tilde{y}_n - C(\mathrm{t})\tilde{y}_n^2\right]d\xi \tag{12}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{y}_n$is a restricted variation, that is, $\delta\tilde{y}_n = 0$. The successive approximation $y_{n+1}, n \geq 0$ of the solution $\mathrm{y(t)}$will be readily obtained upon using Lagrange's multiplier, and by using any selective function y_0. The initial value $\mathrm{y}(0)$and $\mathrm{y}_t(0)$are usually used for selecting the zerothapproximation y_0. To calculate the optimal value of λ, we have

$$\delta y_{n+1} = \delta y_n + \delta\int_0^t \lambda(\xi)\frac{dy_n}{d\xi}d\xi = 0 \tag{13}$$

This yields the stationary conditions

$$\lambda'(\xi) = 0$$
$$1 + \lambda(\xi) = 0$$

which gives

$$\lambda = -1 \tag{14}$$

Substituting this value of Lagrangian multiplier in (11), we get the following iteration formula

$$y_{n+1} = y_n - I^{\alpha}\left[\frac{d^{\alpha}y_n}{d\xi^{\alpha}} - \mathrm{A(t)} - \mathrm{B(t)}y_n - C(\mathrm{t})y_n^2\right] \tag{15}$$

and finally the exact solution is obtained by

$$\mathrm{y(t)} = \lim_{n\to\infty} y_n(t) \tag{16}$$

4. Padé and restrictive Padé

The Padé approximants are a particular type of rational fraction approximation to the value of the function [29-30].

The Padé approximant often given as:

$$H(s) = \frac{A(s)}{B(s)} \tag{17}$$

The Padé approximation can be written in the form

$$PA[M/N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i x^i}{1 + \sum_{i=1}^{N} b_i x^i} \tag{18}$$

where$M$ and$N$ are positive integers

There are $M + 1$ independent numerator coefficients and N denominator coefficients making $M + N + 1$ unknown coefficients.

The $M + N + 1$ unknown suggests that normally the $PA\ [M/N]$ ought to fit the power series $f(x) = \sum_{i=0}^{\infty} c_i x^i$.

Ismail et al. [31-35] applied Restrictive Padé approximation to solve many differential equations. Function approximation that meet Taylor approximation was done by Ismail et al. in [36] .

The restrictive Padé approximation is a rational function in the form:

$$RPA[M + \alpha/N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i x^i + \sum_{i=1}^{\alpha} \varepsilon_i x^{M+i}}{1 + \sum_{i=1}^{N} b_i x^i} \tag{19}$$

where the positive integer α does not exceed the degree of the numerator, $\alpha = 0(1)n$ Such that

$$f(x) = RPA[M + \alpha/N]_{f(x)}(x) + O(X^{M+N+1}) \tag{20}$$

5. Steps of Solution using Vim -restrictive Padé

The VIM gives a semi analytical solution in series form the accuracy of this method depend on two main restrictions the first is the order of series that main when we stop the iteration and make the truncation the second is the region of convergence where the solution of VIM gives high accurate at t=0 which meet the talor series expantion of the exact solution but it has very slow convergence rate in the wider region , for this reasons many researchers make some modifications on VIM [37].

We present a modification of VIM by using the Pade' approximation and then apply this modification to solve the fractional Riccati differential equations. When we obtain the truncated series solution of order at least $L + M$ in t by VIM, we will use it to obtain Pade' approximation $PA[L/M](t)$ for the solution $y(t)$,then we use this series to calculate the restrictive Pade' approximation

Algorithm

Step 1 Apply the VIM to solve our Equation :

Using equation (9) to find the iterative process for equation (1) after that construct the correction function (11) then identify the Lagrangian multiplier, finally we have the iteration formula (15)

Step 2 Truncate the obtained sequence solution by using VIM:

The approximate solution can be obtained as series solution of n degree $y(t) \cong y_n(t)$ for n=1,2,3,..

Step 3 Find the Pade' approximation of the previous step:

We find the Pade' approximation using the series solution that given by VIM

Step 4 Find the restrictive Pade' approximation of the step3:

We find the Pade' approximation using the series solution that given by VIM

Note: in case of fractional order series we make transformation in order to achieve step 3 and step 4 then we reverse the transformation.

6. Numerical example

Example: Consider the following fractional riccati equation

$$\frac{d^\alpha y}{dt^\alpha} = 2y(t) - y^2(t) + 1, \qquad 0 < \alpha \le 1$$

Subject to initial condition

$$y(0) = 0$$

The exact solution, when $\alpha = 1$, is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$$

By starting with $y_0(t) = \frac{t^\alpha}{\Gamma[1+\alpha]}$, then by applying the iteration formulation we can obtain directly the other components as

$$y_1(t) = \frac{x^\alpha}{\Gamma[1+\alpha]} - x^{1+\alpha}\left(\frac{x^\alpha}{(1+2\alpha)\Gamma[1+\alpha]^2} - \frac{2}{\Gamma[2+\alpha]}\right)$$

$$\begin{aligned} y_2(t) = -t^2 &- \frac{t^{3+4\alpha}}{(1+2\alpha)^2(3+4\alpha)\Gamma[1+\alpha]^4} - \frac{t^{1+2\alpha}}{(1+2\alpha)\Gamma[1+\alpha]^2} + \frac{t^\alpha}{\Gamma[1+\alpha]} \\ &- t^{1+\alpha}\left(\frac{t^\alpha}{(1+2\alpha)\Gamma[1+\alpha]^2} - \frac{2}{\Gamma[2+\alpha]}\right) + \frac{2t^{2+3\alpha}(1+\alpha)\big(3(1+\alpha)^2 + t(4+6\alpha)\big)}{3(2+7\alpha+6\alpha^2)\Gamma[2+\alpha]^3} \\ &- \frac{t^{2+2\alpha}(9+21\alpha+10\alpha^2+t(4+8\alpha))}{(3+8\alpha+4\alpha^2)\Gamma[2+\alpha]^2} + \frac{2t^{1+\alpha}}{\Gamma[2+\alpha]} + \frac{2^{1+2\alpha}t^{2+\alpha}\Gamma[\frac{1}{2}+\alpha]}{\sqrt{\pi}\Gamma[3+\alpha]^2} \\ &+ \frac{34^\alpha t^{2+\alpha}\alpha\Gamma[\frac{1}{2}+\alpha]}{\sqrt{\pi}\Gamma[3+\alpha]^2} + \frac{4^\alpha t^{2+\alpha}\alpha^2\Gamma[\frac{1}{2}+\alpha]}{\sqrt{\pi}\Gamma[3+\alpha]^2} + \frac{4t^{2+\alpha}}{\Gamma[3+\alpha]} \end{aligned}$$

At $\alpha = 1$

$$y_1(t) = t + t^2 - \frac{t^3}{3}$$

$$PA\begin{bmatrix}2\\2\end{bmatrix}=\frac{x}{1-x+\frac{2x^2}{3}},$$

$$RPA\begin{bmatrix}2\\2\end{bmatrix}=\frac{x-0.10256410256410256x^2}{1.0-1.102564102564102 6x+0.66667x^2}$$

$$y_2(t)=t+t^2+\frac{t^3}{3}-\frac{2t^4}{3}-\frac{t^5}{15}+\frac{t^6}{9}-\frac{t^7}{63}$$

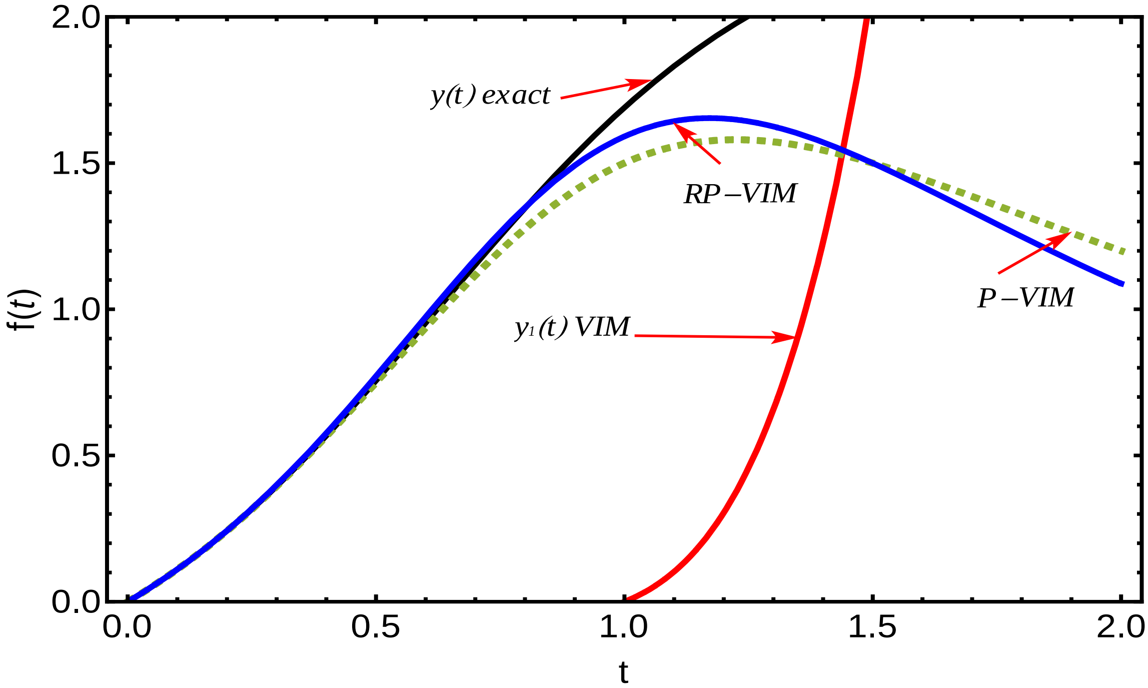

Figure 1: Comparison between exact solution, VIM, Padé -VIM and Restrictive Padé -VIM for $y_1(t)$ at $\alpha=1$

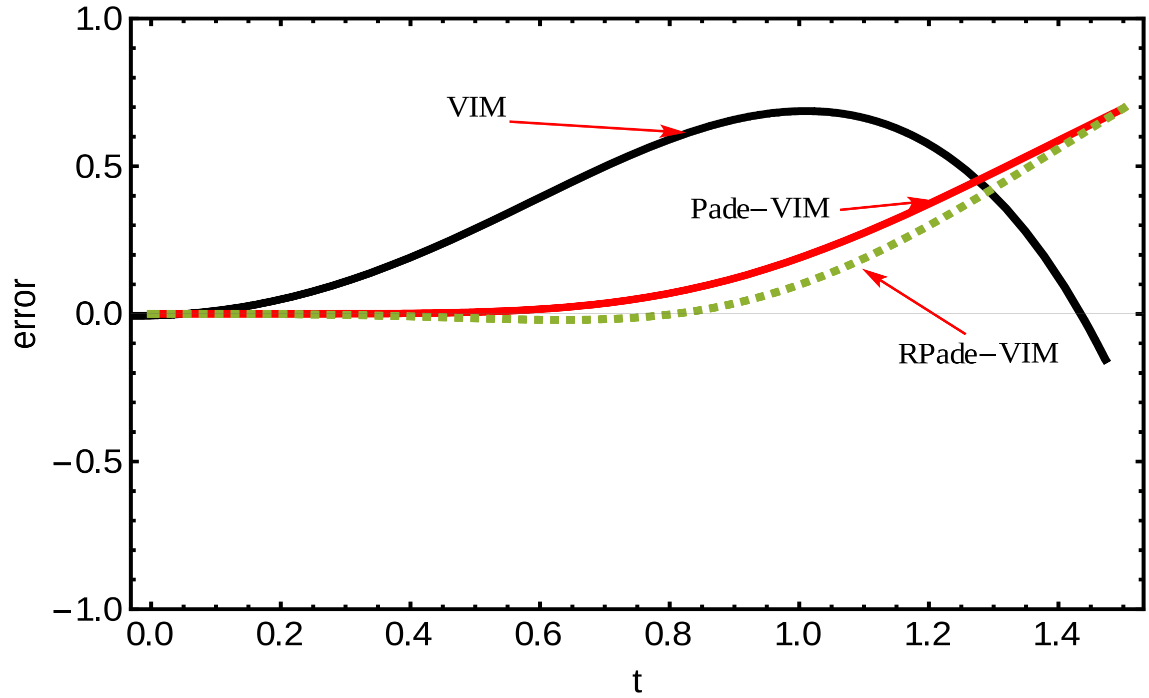

Figure 1: Error between solution, VIM, Padé -VIM and Restrictive Padé -VIM for $y_1(t)$ at $\alpha=1$

At $\alpha=0.5$

$$y_1(t)=1.1283791670955126t^{0.5}+1.5045055561273502t^{1.5}-0.6366197723675813t^2$$

For integer power to approximate Padé and restrictive Padé Let $t^{\frac{1}{2}} = x$, then

$$y_1(t) = 1.1283791670955126\,x + 1.5045055561273502\,x^3 - 0.6366197723675813\,x^4$$

$$PA[2/2] = \frac{1.1283791670955126x + 0.4774648292756859x^2}{1 + 0.4231421876608171x - 1.3333333333333335x^2}$$

$put \quad x = t^{\frac{1}{2}}$

Calculating the [2/2] Pade´ approximants and recalling that x = t1/2, we get

$$PA[2/2] = \frac{1.1283791670955126\,t^{\frac{1}{2}} + 0.4774648292756859\,\mathrm{t}}{1 + 0.4231421876608171\,t^{\frac{1}{2}} - 1.3333333333333335\,\mathrm{t}}$$

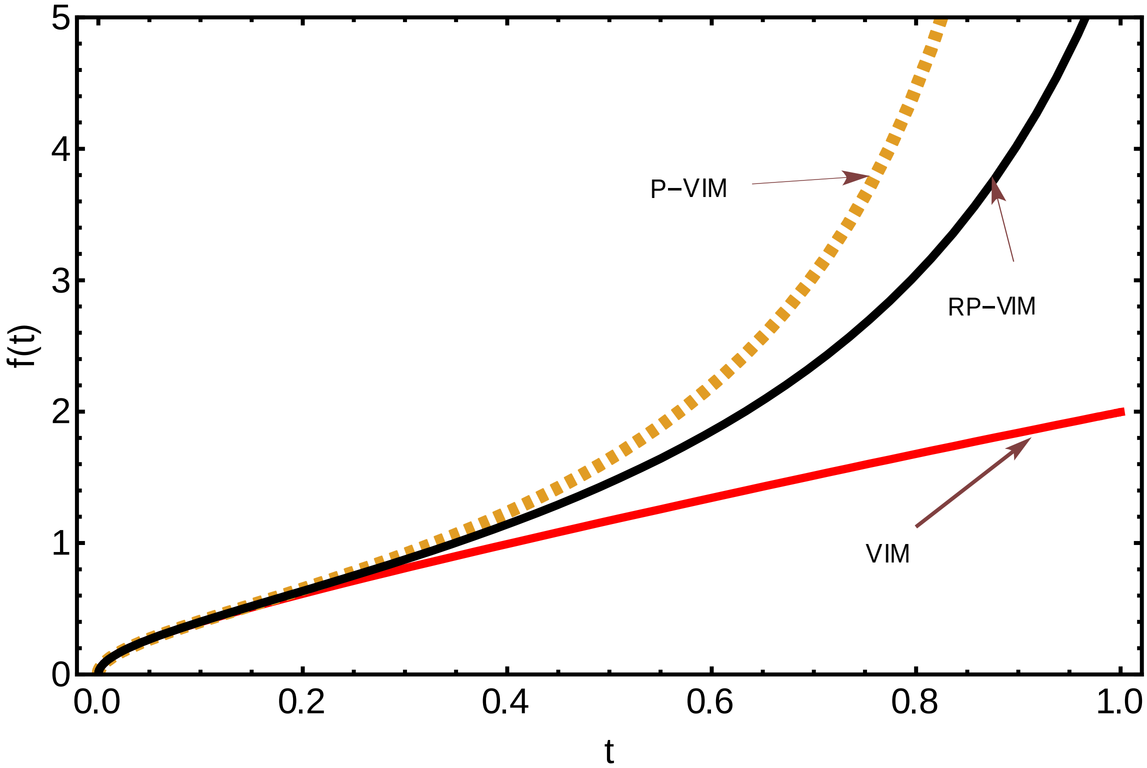

Figure 3: Comparison between VIM,Padé -VIM and Restrictive Padé -VIM for $y_1(t)$ at $\alpha = 0.5$

$$y_2(t) = 1.1283791670955126t^{0.5} + 3.0090111122547007t^{1.5} - t^2 - 1.2732395447351625t^2 + 1.5867236387689023t^{2.5} - 1.5561816657874212t^3 + 0.4104848505718093t^{3.5} - 0.5658842421045168t^4 + 0.42568799318558t^{4.5} - 0.0810569469138702t^5$$

7. Results and Conclusion

Most of engineering applications can be model as rational function called transfer function ,this function describe the relation between input and output this is the main reason that Padé approximation gives more accurate result than semi analytical solutions.

From figures of both result and error we can claim that Padé -VIM is more better than VIM but restrictive Padé -VIM is the best result and less error

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