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Research Article

6-Perfect Number

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Abstract Let $\delta(a)$ denote the sum of all divisors of a positive integer a and let n be a given positive integer. If a positive integer x satisfies $\delta(x)+\delta(nx)=2(n+1)x$, then x is called n—Perfect number. In this paper we investigate the existence of 6-Perfect number by using the basic properties of $\delta(a)$ and prove that 13 is a unique 6-Perfect number.

Keywords Constraints and perfect number; n-Perfect number; 6-Perfect number

1. Introduction

Let N be a set of positive integers and let $\delta(a)$ denote the sum of all divisors of a positive integer a. In number theory the properties of $\delta(a)$ have been received an extensive concern for a long time [1,2,3,4]. For example, if a positive integer x satisfies $\delta(x)=2x$, then x is a perfect number. So far, it still remains a difficult problem far from solved that whether there exist one or more odd perfect numbers or not.

If a integer point (x,y) in elliptic curve (1.1) satisfies $y=0$, then it is a trivial integer point, otherwise a nontrivial integer point. It is obvious that there only exists a trivial integer point $(x,y)=(0,0)$ in elliptic curve (1.1). If a integer point (x,y) in elliptic curve (1.1) is a nontrivial integer point, then so is $(x,-y)$, and they , denoted $(x,\pm y)$ with $y>0$, are together referred to as a pair of nontrivial integer points in elliptic curve (1.1). Let s be a nonnegative integer and let a be a positive integer with $a>1$. In this paper we determine all nontrivial integer points in elliptic curve (1.1) for the case where k is a positive odd number, in other words, we prove the following theorem.

Theorem For a positive integer k, there only exist the following kinds of nontrivial integer points in elliptic curve (1.1):

(I) $p=2, k=4s+3, (x,\pm y)=(2^{2s},\pm 2^{3s}\cdot 3)$ *and* $(2^{2s+3},\pm 2^{3s+3}\cdot 3)$。

(II) $p=3, k=1, (x,\pm y)=(1,\pm 2),(3,\pm 6)$ *and* $(12,\pm 42)$.

(III) $p=3, k=4s+5, (x,\pm y)=(3^{2s}\cdot 121,\pm 3^{3s}\cdot 1342),(3^{2s+2},\pm 3^{3s+3}\cdot 2),(3^{2s+3},\pm 3^{3s+4}\cdot 2)$ and $(3^{2s+3}\cdot 4,\pm 3^{3s+4}\cdot 14)$.

(IV) $p=2a^2+1, k=4s+1, (x,\pm y)=(p^{2s}a^2,\pm p^{3s}a(a^2+1))$.

(V) $p>3, k\equiv n(\bmod 4), (x,\pm y)=(p^{(k+n)|2}Y^2,\pm p^{(3k+n)|4}XY)$, *where* (X,Y,n) is a solution of the equation

$$X^2-p^nY^4=1, X,Y,n\in N,\quad 2\nmid n。\qquad (1.2)$$

Let $N(P^K)$ denote the pair number of nontrivial integer points $(x,\pm y)$ in elliptic curve (1.1). According to the above theorem we can obtain that the following upper bound of $N(P^K)$.

Corollary For a positive odd k, when $p=2$,

$$N(2^k)=\begin{cases}2, & if \quad k\equiv 3(\mathrm{mod}\, 4),\\ 0. & otherwise\end{cases}$$

When p is an odd number,

$$N(p^k)\begin{cases}=4, & if \quad p=3, k\geq 5 \; and \quad k\equiv 1(\mathrm{mod}\, 4),\\ =3, & if \quad p=3 \quad and \quad k=1,\\ \leq 2, & otherwise。\end{cases}$$

2. Lemma

Let D be a non-square positive integer. It follows from Theorem 10.9.1 and 10.9.2 of [5] that the equation

$$U^2-DV^2=1 \qquad U,V\in\mathbb{N} \tag{2.1}$$

has solutions (u,v) and has a unique solution (u_1,v_1) such that $u_1+v_1\sqrt{D}\leq u+v\sqrt{D}$ where (u,v) is an arbitrary solution of (2.1). Therefore, (u_1,v_1) is the minimal solution of (2.1)

Lemma 2.1 The equation

$$X^2-DY^4=1, \quad X,Y\in\mathbb{N} \tag{2.2}$$

has at most two solutions. If it has indeed two solutions (X_1,Y_1) and (X_2,Y_2) such that $X_1<X_2$, then when $D\neq 1785$ or 28560, we have that

$$(X_1,Y_1^2)=(u_1,v_1)\,,\quad (X_2,Y_2^2)=(u_1^2+Dv_1^2,2u_1v_1)\,, \tag{2.3}$$

where (u_1,v_1) is the minimal solution of (2.1).

Proof See Lemma 2 of [6].

Lemma 2.2 For a given integer $m\in\{1,3\}$, if $p=2$, then the equation

$$X^2-p^mY^4=1, \quad X,Y\in\mathbb{N} \tag{2.4}$$

has a solution (X,Y)=(3，1) only if $m=3$. If $p=3$, equation (2.4) has two solutions (X,Y)=(2，1) and （7，2）only if $m=1$. If $p>3$, then equation (2.4) has at most a solution (X,Y).

Proof By the fact that when $D\leq 400$ all solutions of equation (2.2) of [7] can be obtained, we can deduce that this lemma holds for the case of $p\leq 3$. When $p>3$, since $p^m\neq 1785$ or $p^m\neq 28560$, it follows from Lemma 2.1 that if equation(2.4) has two solutions (X_1,Y_1) and (X_2,Y_2) such that $X_1<X_2$, these solutions must satisfies (2.3) where (u_1,v_1) is the minimal solution of

$$U^2-p^mV^2=1 \qquad U,V\in\mathbb{N} \tag{2.5}$$

From (2.3) we know that $v_1=Y_1^2$ and

$$2u_1v_1=2u_1Y_1^2=Y_2^2\;, \tag{2.6}$$

which means that

$$u_1=2a^2, Y_2^2=2aY_1, a\in\mathbb{N}, \tag{2.7}$$

and from (2.5) and (2.7) we also know that

$$u_1^2 - p^m v_1^2 = 4a^4 - p^m Y_1^4 = 1 \quad (2.8)$$

Since p is an odd prime number and $\gcd(2a^2+1, 2a^2-1)=1$, it follows from (2.8) that

$$2a^2+\lambda = b^4, 2a^2-\lambda = p^m c^4, Y_1 = bc, \lambda \in \{\pm 1\}, b, c \in \mathbb{N} \quad (2.9)$$

However, we can deduce from [8] that the first equality in (2.9) does not hold for the case of $\lambda = 1$, infer from [9] that when $\lambda = -1$ the first equality in (2.9) holds only if $a = b = 1$, and thus obtain from the second equality in (2.9) that $p^m = 3$ at this moment, which contradicts $p > 3$. Therefore equation (2.5) has at most a solution. The proof is completed.

Lemma 2.3 The equation

$$1+2X^2 = Y^n, X, Y, n \in \mathbb{N}, \quad n>1, 2\nmid n \quad (2.10)$$

has only a solution $(X,Y,n)=(11,3,5)$.

Proof See [10].

Lemma 2.4 The equation

$$X^2 - Y^4 = p^n, X, Y, n \in \mathbb{N}, \quad \gcd(X,Y)=1, \quad 2\nmid n \quad (2.11)$$

has only three solutions $(p,n,X,Y)=(2,3,3,1),(3,5,122,11)$ and $(2a^2+1, 1, a^2+1, a)$, where a is a positive integer.

Proof Let (p,n,X,Y) be a solution of equation (2.1). when $p=2$, since X and Y are coprime positive odd numbers , we obtain that $\gcd(X+Y^2, X-Y^2)=2$, and also obtain that $X+Y^2=2^{n-1}$ and $X-Y^2=2$ from（2.11）. Therefore,

$$X = 2^{n-2}+1, \quad Y^2 = 2^{n-2}-1 \quad (2.12)$$

Due to $Y^2+1 \equiv 2 \pmod 4$, it follows from (2.12) that equation (2.11) has only a solution $(p,n,X,Y)=(2,3,3,1)$.

When p is an odd prime number, since X is relatively prime to Y and X and Y are respectively odd and even, we have that $\gcd(X+Y^2, X-Y^2)=1$. Therefore it is deduced from (2.11) that $X+Y^2=p^n$, $X-Y^2=1$ and

$$2X = p^n+1, \quad 2Y^2 = p^n-1 \quad (2.13)$$

Due to $2\nmid n$, it can follows from Lemma 2.3 and (2.13) that equation (2.11) has only two solutions（3,5,122,11)和 $(2a^2+1, 1, a^2+1, a)$. The proof is completed.

3. Proof

Proof of the above theorem Let $(x, \pm y)$ denote a pair of nontrivial integer points in elliptic curve (1.1). Due to $y>0$, we know that $x>0$ and x can be uniquely expressed as

$$x = p^r - z, r \in \mathbb{Z}, \quad r \geq 0, z \in \mathbb{N}, p \nmid z \quad (3.1)$$

Substituting (3.1) into (1.1) yields that

$$p^r z(p^{2r}z^2 + p^k) = y^2 \quad (3.2)$$

Since k is a positive odd number, $k \neq 2r$. Therefore, all that is required is to consider the following two case:

Case Ⅰ: $k > 2r$

It follows from (3.2) that

$$p^{3r} Z(Z^2 + p^{k-2r}) = y^2 \quad (3.3)$$

Due to $p \nmid z$, we have that $p \nmid z^2 + p^{k-2r}$ and $\gcd(z, z^2 + p^{k-2r}) = 1$, and thus derive from (3.3) that

$$r = 2s, z = f^2, z^2 + p^{k-2r} = g^2, y = p^{3s} fg, s \in \mathbb{Z}, s \geq 0, f, g \in \mathbb{N}, \gcd(f, g) = 1 \quad (3.4)$$

which gives that

$$g^2 - f^4 = p^{k-4s} \quad . \quad (3.5)$$

When $p = 2$, it is inferred from Lemma2.4 and (3.5) that $k - 4s = 3, f = 1$ and $g = 3$, and thus is inferred from (3.4) that

$$p = 2, \quad k = 4s + 3, \quad (x, \pm y) = (2^{2s}, \pm 2^{3s} \cdot 3) \quad (3.6)$$

When p is an odd number, it follows from Lemma2.4, (3.4) and (3.5)

$$p = 3, \quad k = 4s + 5, \quad (x, \pm y) = (3^{2s} \cdot 121, \pm 3^{3s} \cdot 1342) \quad (3.7)$$

and

$$p = 2a^2 + 1, \quad k = 4s + 1, \quad (x, \pm y) = (p^{2s} a^2, \pm p^{3s} a(a^2 + 1)) \quad (3.8)$$

Case II: $k < 2r$

It follows from (3.2) that

$$p^{r+k} z(p^{2r-k} z^2 + 1) = y^2 \quad (3.9)$$

Since $p \nmid z(p^{2r-k} z^2 + 1)$ and $\gcd(z, p^{2r-k} z^2 + 1) = 1$, it is inferred from (3.9) that

$$r + k = 2t, z = f^2, p^{2r-k} z^2 + 1 = g^2, y = p^t fg, \ t, f, g \in \mathbb{N}, \gcd(f, g) = 1 \quad (3.10)$$

which yields that

$$g^2 - p^{2r-k} f^4 = 1 \quad (3.11)$$

The above equality (3.11) also means that equation (1.2) has a solution (X, Y, n) such that

$$(X, Y, n) = (g, f, 2r - k) \quad (3.12)$$

Since it is deduced from the first equality in (3.10) that r is a positive odd number, we have $r = 2s + 1$ where s is a nonnegative integer. Since it is also deduced from (3.12) that $n = 2r - k$, we obtain from $n \equiv 2r + k \equiv 2 - k (\bmod 4)$ that $k \equiv n (\bmod 4)$. Therefore, according to Lemma2.2, (3.10) and (3.11), we know that there only exist these integer points

$$p = 2, \quad k = 4s + 3, \quad (x, \pm y) = (2^{2s+3}, \pm 2^{3s+3} \cdot 3) \quad (3.13)$$

$$p = 3, \quad k = 4s + 1, \quad (x, \pm y) = (3^{2s+1}, \pm 3^{3s+1} \cdot 2) \quad (3.14)$$

$$p = 3, \quad k = 4s + 1, \quad (x, \pm y) = (3^{2s+1} \cdot 4., \pm 3^{3s+1} \cdot 14) \quad (3.15)$$

and integer points of (V).

In conclusion, several kinds of integer points mentioned above are obtained by the following way. These integer points of (I) are generated by combining (3.6) and (3.13), these integer points of (II) are obtained by taking $a = 1$ and $s = 0$ in (3.8) and $a = 1$ and $s = 0$ in (3.15) and combing them, these integer points of (III) are gained by taking $a = 1$ and $s \geq 1$ in (3.8), taking $s \geq 1$ in (3.14) and (3.15) and combining them with (3.7), and these integer points of (IV) are obtained by taking $a > 1$ in (3.8). The proof is completed.

The proof of corollary It follows from the foregoing theorem that the corollary holds for the case of $p \le 3$. Let N_4 and N_5 be respectively the pair number of nontrivial integer points of (IV) and (V) in elliptic curve(1.1). Obviously, it follows from the foregoing theorem that

$$N(p^k) = N_4 + N_5 \qquad (3.16)$$

Since p is given, a is given when $p = 2a^2 + 1$, and we have $N_4 \le 1$. Together with the fact that k is given, we obtain $N_5 \le 1$ from Lemma 2.2, and thus obtain $N(p^k) \le 2$ from (3.16). The proof is completed.

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