Available online www.jsaer.com

Journal of Scientific and Engineering Research, 2016, 3(4):539-542

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Convergence and Stability Properties of an Inverse Polynomial Scheme for the Solution of Initial Value Problems

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Abstract In this paper, we investigate the convergence and stability properties of a one-step method for numerical scheme of ordinary differential equations is. We proved that the one step inverse polynomial method is stable and convergent.

Keywords Initial value problem, stability, one step, convergent, consistent.

Introduction

We shall consider the initial value problem (I. V. P) of the form

$$y' = f(x,y), y(0) = y_{0,} x \in [a,b], y, f \in R \qquad (1.1)$$

Numerical analysis naturally finds application in all fields of Engineering and the Physical sciences, but in the 21st century also, the life sciences and even the arts have adopted elements of scientific computations [1].

Equation of the form (1) arises in a variety of disciplines. These include Physics (both pure and applied), Engineering, Medicine and even in social sciences. It is a known fact that some of the formulations of government economic policies are based on equation of the form (1). For example, the scientific analysis of population explosion is based on the equation of the same form. (1)

Definition 1.0

A one-step method can be defined as

$$y_{n+1} = y_n + h\Phi(x_n, y_n; h) \quad (1.2)$$

Where $\Phi(x_n, y_n; h)$ can be defined as the increment function.

In this paper, we shall assume that the theoreticalapproximation $y_n + h$ evaluated at $x = x_n + h$ to the exact solution $y(x_n + h)$ to the first order ordinary differential equationbe represented as

$$y(x_{n+1}) = y_n[e^p + \sum_{j=1}^{k} b_j x_n^j]^{-1} \quad (1.3)$$

Where e^p is the exponential of p (setting p=0) and the parameters $b_j s'$ are to be determine from the non-linear equations

With the assumption that $y(x)$ approximate (3), we obtain a one-step method of the form

$$y_{n+1} = \frac{6y_n^4}{6y_n^3 - 6hy_n^2 y_n' + 3h^2 y_n[2(y_n')^2 - y_n y_n''] + h^3[6y_n y_n' y_n'' - 6(y'n)^3 - y_n''' y_n^2]} \qquad (1.4)$$

Equation (4) will be the major reference of this work. In short, we shall show that the method represented by (4) is stable and convergent. There had been numerous methods developed to solve initial value problems in ordinary differential equations.

Numerical analysis can be explained as a study of algorithm that uses numerical approximation for the problems of mathematical analysis. It is concerned with obtaining approximate solutions while maintaining reasonable bounds on errors.

It is possible to clarify numerical analysis as an art or a science. Science in the sense in that it has to do with the generation of algorithms and art in the sense that it is also concerned withevaluation/assessment of these algorithms with the objective to identify the best for a particular problem.

Definition 1.1. Any algorithm for solving a differential equation in which the approximation y_{n+1} to the solution at the point x_{n+1} can be calculated if only x_n, y_n and h is known as a one-step method.

It is a common practice to write the functional dependence, y_{n+1}, on the quantities x_n, y_n and h in the form $y_{n+1} = y_n + h\emptyset(x_n, y_n; h)$.

Theorem 1.0 [1]

Let the increment function of the scheme defined above be continuous, jointly as a function of its arguments in the region defined by

$x \in [a,b]$ *and* $y \in (-\infty,\infty)$; $0 \le h \le h_0$, where $h_0 > 0$, and let there exists a constant L such that $|\emptyset(x_n, y_n^*; h) - \emptyset(x_n, y_n; h)| \le L|y_n^* - y_n|$

For all $(x_n, y_n; h)$ and $(x_n, y_n^*; h)$ in the region just define.

Then the relation $(x_n, y_n \;\; ; 0) = (x_n, y_n \;\;)$ is a necessary and sufficient condition for the convergence of the new scheme defined above in (1.4)

The Consistency Property of The Scheme

The conventional one step integrator for the Initial value problem (1.1) is generally described according to Lambert (1963) [2] as

$y_{n+1} = y_n + h\emptyset(x_n, y_n, h)$(2.1)

By subtracting y_n from both sides of (2.1) we obtain

$$y_{n+1} - y_n = h\emptyset(x_n, y_n, h) \quad (2.2)$$

Divide both sides of (1.5) by h, we have

$$\frac{y_{n+1} - y_n}{h} = \frac{h\emptyset(x_n, y_n, h)}{h}$$

$$\frac{y_{n+1} - y_n}{h} = \emptyset(x_n, y_n, h) \quad (2.3)$$

If $\emptyset(x_n, y_n, h) = f(x, y)$, (2.4)

$\emptyset(x_n, y_n; 0) = f(x, y)$ (2.5)

Therefore a consistent method has order of at least one. We say our new numerical method is consistence since equation (1.4.) reduces to (2.4) when $h = 0$, therefore, we say that the method is consistent

Proof: Given that the given integrator formula(scheme) is consistent with the Initial value problem under consideration.

to show this with respect to the scheme derived above, subtract y_n from both sides of (2.5)

$$y_{n+1} - y_n = \frac{6y_n^4}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3(y_n')^3 - h^3y_n'''y_n^2} - \frac{y_n}{1} \quad (2.6)$$

$$y_{n+1} - y_n = \frac{6y_n^4 - 6y_n^4 + 6hy_n^3y_n' - 6h^2y_n^2(y_n')^2 + 3h^2y_n^3y_n'' - 6h^3y_n^2y_n'y_n'' + 6h^3y_n^3(y_n') + h^3y_n'''y_n^3}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_n'y_ny_n'' - 6h^3(y_n')^3 - h^3y_n'''y_n^2} \quad (2.7)$$

$$y_{n+1} - y_n = \frac{h[6y_n^3y_n' - 6hy_n^2(y_n')^2 + 3hy_n^3y_n'' - 6h^2y_n^2y_n'y_n'' + 6h^2y_n^3(y_n') + h^2y_n'''y_n^3]}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_ny_n'y_n'' - 6h^3(y_n') - h^3y_n'''y_n^2} \quad (2.8)$$

Divide both sides of (2.8) by h,

$$\frac{y_{n+1} - y_n}{h} = \frac{6y_n^3y_n' - 6hy_n^2(y_n')^2 + 3hy_n^3y_n'' - 6h^2y_n^2y_n'y_n'' + 6h^2y_n^3(y_n') + h^2y_n'''y_n^3}{6y_n^3 - 6hy_n^2y_n' + 6h^2y_n(y_n')^2 - 3h^2y_n^2y_n'' + 6h^3y_ny_n'y_n'' - 6h^3(y_n') - h^3y_n'''y_n^2} \quad (2.9)$$

As h tends to zero yields

$$\frac{y_{n+1} - y_n}{h} = \frac{6y_n^3y_n'}{6y_n^3}$$

$$\frac{y_{n+1} - y_n}{h} = y_n' \quad (2.10)$$

Which implies that (2.3) is satisfied and thus the scheme (1.4) is consistent.

The Stability Property of The Scheme

THEOREM 3.1 (Fatunla, 1988) [3]

Let $y_n = y(x_n)$ and $p_n = p(x_n)$ denote two different numerical solution of differential equation with the initial conditions specified as

$y(x_0) = \zeta$ and $p(x_0) = \zeta^*$ respectively, such that $|\zeta - \zeta^*| < \varepsilon$, $\varepsilon > 0$.

If the two numerical estimates are generated by the integration scheme, we have:

$$y_{n+1} = y_n + h\emptyset(x_n, y_n; h) \quad (3.1)$$

$$p_{n+1} = p_n + h\emptyset(x_n, p_n; h) \quad (3.2)$$

The condition that

$|y_{n+1} - p_{n+1}| \le k|\zeta - \zeta^*|$

Is the necessary and sufficient condition that our new method is stable and convergent.

Proof:

We use the general form of the scheme to investigate the stability property

$$y_{n+h} = \frac{y_{n+h-1}}{(e^p + \sum_{j=1}^{k} b_j x_{n+h}^j)} \quad (3.3)$$

The theoretical solution y(x) is given as

$$y(x_{n+h}) = \frac{y(x_{n+h-1})}{(e^p + \sum_{j=1}^{k} b_j x_{n+h}^j)} + T_{n+h} \quad (3.4)$$

By subtraction

$$y(x_{n+h}) - y_{n+h} = \frac{y(x_{n+h-1})}{(e^p + \sum_{j=1}^{k} b_j x_{n+h}^j)} - \frac{y_{n+h-1}}{(e^p + \sum_{j=1}^{k} b_j x_{n+h}^j)} + T_{n+h} \quad (3.5)$$

But the globalization error associated with general one-step scheme (1.4) is given by

$$e_{n+h} = y_{n+h} - y(_{n+h}) \quad (3.6)$$

Now by adopting (3.6) on (3.5) and simplifying, we obtain

$$e_{n+h} = \frac{e_{n+h-1}}{(e^p + \sum_{j=1}^{k} b_j x_{n+h}^j)} + T_{n+h} \quad (3.7)$$

But since p=0 is a constant, then e^p=1

Hence,

$$e_{n+h} = \frac{e_{n+h-1}}{1 + \sum_{j=1}^{k} b_j x_{n+h}^j} + T_{n+h} \quad (3.8)$$

Taking the modulus of both sides, yields $\left| \frac{1}{1+\sum_{j=1}^{k} b_j x_{n+h}^j} \right| = \frac{1}{1+\sum_{j=1}^{k} b_j x_{n+h}^j}$

By setting, Q=$| 1 + \sum_{j=1}^{k} b_j x_{n+h}^j |$, we have

$$\left| \frac{1}{1 + \sum_{j=1}^{k} b_j x_{n+h}^j} \right| = \frac{1}{Q} = M$$

Then,

$$|e_{n+h}| \leq M|e_{n+h}| + |T_{n+h}|$$

Let T= sup(T_{n+h}) and M<1 similarly by setting $E_{n+h} = \sup e_{n+h,}$

Then, the inequality modifies into 0<n<∞

$$E_{n+h} \leq ME_{n+h-1} + T$$

Hence for h=1, we have

$$E_{n+1} \leq ME_n + T$$

For h=2,

$$E_{n+2} \leq M^2 E_n + MT + T \quad (3.9)$$

By following this trend , it could be seen that

$$E_{n+h} \leq M^k E_{n+h-1} + \sum_{r=0} M^r T \quad (3.10)$$

Since M<1, then as n tends to infinity, $E_{n+h} \longrightarrow 0$.

Convergence Property of the Scheme

Having tested for the consistency and stability of the method, we can conclude that the convergence property is also satisfied.

THEOREM

Let $y_n = y(x_n)$ and $l_n = l(x_n)$ denote two different numerical solutions of differential equation (1.1) with initial conditions specified as $y(x_0) = \mu \; and \; l(x_0) = \mu *$ respectively, such that $|\mu - \mu *| < \varepsilon$, $\varepsilon > 0$. If the two numerical estimates are generated by the integration scheme (1.4)

We have

$$y_{n+1} = y_n + h\emptyset(x_n, y_n, h)$$
$$l_{n+1} = l_n + h\emptyset(x_n, l_n, h)$$

The condition that

$|y_{n+1} - l_{n+1}| \leq k|\mu - \mu *|$is the necessary and sufficient condition that the method/scheme is stable and convergent.

PROOF

From (1.4)

$$y_{n+1} - y_n = \frac{6y_n^4 - 6y_n^4 + 6hy_n^3 y_n' - 6h^2 y_n^2 (y_n')^2 + 3h^2 y_n^3 y_n'' - 6h^3 y_n^2 y_n' y_n y_n'' + 6h^3 y_n (y_n') + h^3 y_n''' y_n^3}{6y_n^3 - 6hy_n^2 y_n' + 6h^2 y_n (y_n')^2 - 3h^2 y_n^2 y_n'' + 6h^3 y_n' y_n y_n'' - 6h^3 (y_n')^3 - h^3 y_n''' y_n^2}$$

Adding y_n to both sides

We have that,

$$y_{n+1} = \frac{y_n}{1} + \frac{h[6y_n^3 y_n' - 6hy_n^2 (y_n')^2 + 3hy_n^3 y_n'' - 6h^2 y_n^2 y_n' y_n'' + 6h^2 y_n (y_n') + h^3 y_n''' y_n^3}{6y_n^3 - 6hy_n^2 y_n' + 6h^2 y_n (y_n')^2 - 3h^2 y_n^2 y_n'' + 6h^3 y_n' y_n y_n'' - 6h^3 (y_n') - h^3 y_n''' y_n^2}$$

$$y_{n+1} = \frac{6y_n^4 - 6h^3y_n^3y_n^{'} + 6h^2y_n^2(y_n^{'})^2 - 3h^2y_n^3y_n^{''} + 6h^3y_n^2y_n^{'}y_n^{''} - 6h^3y_n(y_n^{'}) - h^3y_n^{'''}y_n^3 + h[6y_n^3y_n^{'} - 6hy_n^2(y_n^{'})^2 + 3hy_n^3y_n^{''} - 6h^2y_n^2y_n^{'}}{6y_n^3 - 6hy_n^2y_n^{'} + 6h^2y_n(y_n^{'})^2 - 3h^2y_n^2y_n^{''} + 6h^3y_n^{'}y_ny_n^{''} - 6h^3(y_n^{'}) - h^3y_n^{'''}y_n^2}$$

$$= \qquad y_{n+1} = \frac{6y_n^4 - 6h^3y_n^3y_n^{'} - 6h^3y_n(y_n^{'}) - h^3y_n^{'''}y_n^3 + 6hy_n^3y_n^{'} + 6h^3(y_n^{'})^3}{6y_n^3 - 6h^3y_n^2y_n^{'} + 6h^2y_n(y_n^{'})^2 - 3h^2y_n^2y_n^{''} + 6h^3y_n^{'}y_ny_n^{''} - 6h^3(y_n^{'}) - h^3y_n^{'''}y_n^2}$$

Similarly,
From(2.1)

$$l_{n+1} = l_n + h\emptyset(x_n, l_n, h)$$
$$l_{n+1} - l_n = h\emptyset(x_n, l_n, h)$$
$$l_{n+1} - l_n = \frac{6l_n^4 - 6l_n^4 + 6hl_n^3l_n^{'} - 6h^2l_n^2(l_n^{'})^2 + 3h^2l_n^3l_n^{''} - 6h^3l_n^2l_n^{'}l_n^{''} + 6h^3l_n(l_n^{'}) + h^3l_n^{'''}l_n^3}{6l_n^3 - 6hl_n^2l_n^{'} + 6h^2l_n(l_n^{'})^2 - 3h^2l_n^2l_n^{''} + 6h^3l_n^{'}l_nl_n^{''} - 6h^3(l_n^{'})^3 - h^3l_n^{'''}l_n^2}$$

Adding l_n to both sides
We have that,

$$l_{n+1} = \frac{l_n}{1} + \frac{h[6l_n^3l_n^{'} - 6hl_n^2(l_n^{'})^2 + 3hl_n^3l_n^{''} - 6h^2l_n^2l_n^{'}l_n^{''} + 6h^2l_n(l_n^{'}) + h^3l_n^{'''}l_n^3}{6l_n^3 - 6hl_n^2l_n^{'} + 6h^2l_n(l_n^{'})^2 - 3h^2l_n^2l_n^{''} + 6h^3l_n^{'}l_nl_n^{''} - 6h^3(l_n^{'}) - h^3l_n^{'''}l_n^2}$$

$$l_{n+1} =$$

$$l_{n+1} = \frac{6l_n^4 - 6h^3l_n^3l_n^{'} - 6h^3l_n(l_n^{'}) - h^3l_n^{'''}l_n^3 + 6hl_n^3l_n^{'} + 6h^3(l_n^{'})^3}{6l_n^3 - 6h^3l_n^2y_n^{'} + 6h^2l_n(l_n^{'})^2 - 3h^2l_n^2l_n^{''} + 6h^3l_n^{'}l_nl_n^{''} - 6h^3(l_n^{'}) - h^3l_n^{'''}l_n^2} \quad y_{n+1} - l_{n+1} =$$

$$\frac{6y_n^4 - 6h^3y_n^3y_n^{'} - 6h^3y_n(y_n^{'}) - h^3y_n^{'''}y_n^3 + 6hy_n^3y_n^{'} + 6h^3(y_n^{'})^3}{6y_n^3 - 6h^3y_n^2y_n^{'} + 6h^2y_n(y_n^{'})^2 - 3h^2y_n^2y_n^{''} + 6h^3y_n^{'}y_ny_n^{''} - 6h^3(y_n^{'}) - h^3y_n^{'''}y_n^2} - \frac{6l_n^4 - 6h^3l_n^3l_n^{'} - 6h^3l_n(l_n^{'}) - h^3l_n^{'''}l_n^3 + 6hl_n^3l_n^{'} + 6h^3(l_n^{'})^3}{6l_n^3 - 6h^3l_n^2y_n^{'} + 6h^2l_n(l_n^{'})^2 - 3h^2l_n^2l_n^{''} + 6h^3l_n^{'}l_nl_n^{''} - 6h^3(l_n^{'}) - h^3l_n^{'''}l_n^2}$$

As $h \to 0, we\ have$

$$y_{n+1} - l_{n+1} = \frac{6y_n^4}{6y_n^3} - \frac{6l_n^4}{6l_n^3}$$
$$|y_{n+1} - l_{n+1}| = |y_n - l_n|$$
$$|y_{n+1} - l_{n+1}| = 1|y_n - l_n|$$

But since the initial conditions are given as $y(x_0) = \mu\ and\ l(x_0) = \mu *$ respectively, such that $|\mu - \mu *| < \varepsilon$, $\varepsilon > 0$.
Hence,

$$|y_{n+1} - l_{n+1}| = 1|\mu - \mu *|$$

This establishes the convergence property of the scheme.

Conclusion

We have presented a comprehensive detail of the convergence, stability and consistency of a non-linear method, we investigated the convergence and stability properties of the numerical scheme. We also established that the one step inverse polynomial method is stable and convergent

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