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Some Generalised Results for Emanant in Argand Plane

Roshan Lal

Depatment of Mathematics, Govt. Degree College Chaubattakhal Pauri Garhwal, Uttarakhand, India

Abstract Let $p(z)$ be a polynomial of degree n and let α be any real or complex number, then the polar derivative of $p(z)$ denoted by $D_\alpha\, p(z)$, is defined as

$$D_\alpha\, p(z) = n\,p(z) + (\alpha - z)\,p'(z)$$

The polynomial $D_\alpha\, p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ of $p(z)$ in the sense that

$$\underset{\alpha\to\infty}{Lim}\ \frac{D_\alpha\, p(z)}{\alpha} = p'(z)$$

For a polynomial p(z) having all its zeros in $|z|<k$, $k\le 1$ and for all $\alpha_1,\alpha_2,\ldots,\alpha_t,$ $1\le t\le n$ with $|\alpha_1|\ge k$, $|\alpha_1|\ge k$, $|\alpha_2|\ge k$,....., $|\alpha_t|\ge k$, $1\le t<n$, A. Zireh [J. Ineq. and Appl., 2011, 1-9] proved that

$$\max_{|z|=1}\left|D_{\alpha_t}\ldots D_{\alpha_2}D_{\alpha_1}p(z)\right| \ge \frac{n(n-1)\ldots\ldots(n-t+1)}{(1+k)^{tt}}$$
$$\times\left[\{(|\alpha_1|-k)\ldots\ldots(|\alpha_t|-k)\}\max_{|z|=1}|p(z)| + \left\{\begin{matrix}(1+k)^t(|\alpha_1||\alpha_2|\ldots.|\alpha_t|)\\ -\{(|\alpha\ \ |-k)\ldots(|\alpha_t|-k)\}k^{-n}\end{matrix}\right\}\min_{|z|=k}|p(z)|\right].$$

In this paper, we extend this result to the lacunary type of polynomial $p(z)=a_n z^n+\sum_{j=\mu}^{n} a_{n-j}z^{n-j},\ 1\le\mu\le n$, having all its zeros in $|z|<k$, $k\le 1$.

Our results generalize some of the well-known inequalities for the polar derivative of polynomials.

Keywords Polynomials; Polar derivative; Inequalities; Zeros, Maximum modulus.
AMS Subject Classification: 30A10, 30C10, 30D15.

Introduction

Let $p(z)$ be a polynomial of degree n, then according to a famous result known as Bernstein's inequality (for reference see [1]),

$$\max_{|z|=1}|p'(z)| \le n\max_{|z|=1}|p(z)| . \tag{1.1}$$

The result is best possible and equality holds for $p(z)=\lambda z^n$, $\lambda(\neq 0)$ *being a complex number.*

Turan [2] considered the class of polynomials having all the zeros in $|z|\le 1$ and proved the following

Theorem A .*If* $p(z)$ *is a polynomial of degree* n *, having all its zeros in* $|z|\le 1$, *then*

$$\max_{|z|=1}|p'(z)| \ge \frac{n}{2}\max_{|z|=1}|p(z)| . \tag{1.2}$$

The result is sharp and equality in (1.2) *holds for the polynomial* $p(z)=(1+z)^n$.

The following interesting refinement of Theorem A was proved by Aziz and Dawood [3].

Theorem B. *If* $p(z)$ *is a polynomial of degree n, which has all its zeros in* $|z|\le 1$, *then*

$$\max_{|z|=1}|p'(z)| \ge \frac{n}{2}\left\{\max_{|z|=1}|p(z)|+\min_{|z|=1}|p(z)|\right\}. \tag{1.3}$$

Malik [4] obtained the following generalization of (1.2).

Theorem C. *If* $p(z)$ *is a polynomial of degree n, having all its zeros in* $|z|\le k$, $k\le 1$, *then*

$$\max_{|z|=1}|p'(z)| \ge \frac{n}{1+k}\max_{|z|=1}|p(z)| . \tag{1.4}$$

The result is sharp and extremal polynomial is $p(z)=(z+k)^n$.

Inequality (1.4) was generalized by Chan and Malik [5] to the lacunary type of polynomial and proved that if $p(z)=a_n z^n+\sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1\le \mu\le n$, having all zeros in $|z|<k$, $k\le 1$, then

$$\max_{|z|=1}|p'(z)| \ge \frac{n}{1+k^{\mu}}\max_{|z|=1}|p(z)| \tag{1.5}$$

Let $p(z)$ be a polynomial of degree n, and let α be any real or complex number, then the polar derivative of $p(z)$ denoted by $D_\alpha\, p(z)$, is defined as

$$D_\alpha\, p(z)=n\,p(z)+(\alpha-z)p'(z) \tag{1.6}$$

The polynomial $D_\alpha\, p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ of $p(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha\, p(z)}{\alpha}=p'(z) \tag{1.7}$$

The polynomial $D_\alpha\, p(z)$ is called by Laguerre [6] the "emanant" of $p(z)$, by Polya and Szegö [7] the "derivative of $p(z)$ with respect to the point α", and by Marden [8] simply the "polar derivative" of $p(z)$.

It is obviously of interest to obtain estimates concerning growth of $D_\alpha\, p(z)$.

Shah [9] extended Theorem A due to Turan to the polar derivative of polynomial $p(z)$ by proving the following

Theorem D . *If all the zeros of a polynomial* $p(z)$ *of degree* n , *lie in* $|z|\le 1$, *then for every real or complex number* α *with* $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha\, p(z)| \ge \frac{n}{2}(|\alpha|-1)\max_{|z|=1}|p(z)| . \tag{1.8}$$

The result is best possible and extremal polynomial is $p(z)=(z-1)^n$ *with real* $\alpha\ge 1$.

As a refinement of Theorem D, Aziz and Rather [10] also proved the following result.

Theorem E. *If all the zeros of the* n^{th} *degree polynomial* $p(z)$ *lie in* $|z| \leq 1$, *then for every real or complex number* α *with* $|\alpha| \geq 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n}{2}\left\{ \left(|\alpha|-1\right)\max_{|z|=1}|p(z)| + \left(|\alpha|+1\right)\min_{|z|=1}|p(z)| \right\}. \tag{1.9}$$

Govil [11] extended Theorem C to the polar derivative of a polynomial by proving the following result.

Theorem F. *If* $p(z)$ *is a polynomial of degree n having all its zeros in* $|z| < k$, $k \leq 1$, , *then for every real or complex number* α *with* $|\alpha| \geq k$,

$$\max_{|z|=1}|D_\alpha p(z)| \geq n\left(\frac{|\alpha|-k}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{1.10}$$

Next result due to Dewan and Upadhye [12] extends the above result to polar derivative and also generalizes it to lacunary type of polynomial and proved

Theorem G. *If* $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial having all zeros in $|z| < k$, $k \leq 1$, then *for every real or complex number* α *with* $|\alpha| \geq k^\mu$,

$$\max_{|z|=1}|D_\alpha p(z)| \geq n\left(\frac{|\alpha|-k^\mu}{1+k^\mu}\right)\max_{|z|=1}|p(z)|. \tag{1.11}$$

As an extension of of inequality (1.11) Jain [13] proved that if p(z) is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then for all $\alpha_1, \alpha_2, \ldots, \alpha_t,$ $1 \leq t \leq n$, with $|\alpha_1| \geq 1$, $|\alpha_2| \geq 1$, $|\alpha_3| \geq 1$,....., $|\alpha_t| \geq k$, $1 \leq t$ $<n$, we have

$$\max_{|z|=1}\left|D_{\alpha_t}..D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1)\ldots\ldots(n-t+1)}{(2)^{tt}}$$

$$\times\left[\{(|\alpha_1|-1)\ldots\ldots(|\alpha_t|-1)\}\max_{|z|=1}|p(z)| + \left\{\begin{array}{l} 2^t\left(|\alpha_1||\alpha_2|\ldots|\alpha_t|\right) \\ -\{(|\alpha_1|-1)..(|\alpha_t|-1)\}\min_{|z|=1}|p(z)| \end{array}\right\}\right]. \tag{1.12}$$

Inequality (1.12) is generalized by Zireh [14] and proved that if a polynomial p(z) having all its zeros in $|z| < k$, $k \leq 1$, and for all $\alpha_1, \alpha_2, \ldots, \alpha_t,$ $1 \leq t \leq n$ with $|\alpha_1| \geq k$, $|\alpha_1| \geq k$, $|\alpha_2| \geq k$,....., $|\alpha_t| \geq k$, $1 \leq t < n$, then we have

$$\max_{|z|=1}\left|D_{\alpha_t}..D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1)\ldots\ldots(n-t+1)}{(1+k)^{tt}}$$

$$\times\left[\{(|\alpha_1|-k)\ldots\ldots(|\alpha_t|-k)\}\max_{|z|=1}|p(z)| + \left\{\begin{array}{l} (1+k)^t\left(|\alpha_1||\alpha_2|\ldots|\alpha_t|\right) \\ -\{(|\alpha \ |-k)..(|\alpha_t|-k)\}k^{-n}\min_{|z|=k}|p(z)| \end{array}\right\}\right]. \tag{1.13}$$

In this paper, we consider the lacunary type of polynomial $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, and generalize inequality (1.13) for polar derivative of polynomial.

Theorem1. If $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial having all its zeros in $|z| < k$, $k \leq 1$, and for all $\alpha_1, \alpha_2, \ldots, \alpha_t,$ $1 \leq t \leq n$ with $|\alpha_1| \geq k^\mu$, $|\alpha_2| \geq k^\mu$,....., $|\alpha_t| \geq k^\mu$, $1 \leq t < n$, and $|z| = 1$ then we have

$$\left|D_{\alpha_t}..D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1).......(n-t+1)}{\left(1+k^{\mu}\right)^{tt}}$$
$$\times\left[\left\{\left(\left|\alpha_1\right|-k^{\mu}\right).....\left(\left|\alpha_t\right|-k^{\mu}\right)\right\}\max_{|z|=1}|p(z)| + \left\{\begin{array}{l}\left(1+k^{\mu}\right)^t\left(|\alpha_1||\alpha_2|....|\alpha_t|\right) \\ -\left\{\left(\left|\alpha\right|-k^{\mu}\right)..\left(\left|\alpha_t\right|-k^{\mu}\right)\right\}k^{-n}\min_{|z|=k}|p(z)|\end{array}\right\}\right]. \quad (1.14)$$

Equality in (1.14) holds for $p(z)=\left(z-k^{\mu}\right)^n$.

Remark2.If we take $\mu=1$ in above theorem, we get inequality (1.13) due to Zireh [14]. For k=1 above theorem reduces to inequality (1.12) proved by Jain [13].

If we divide the inequality (1.14) by $|\alpha_1\alpha_2.......\alpha_t|$ and letting $|\alpha_1\alpha_2.......\alpha_t|\to\infty$, we get the following

Corollary3. If $p(z)=a_n z^n+\sum_{j=\mu}^{n} a_{n-j}z^{n-j},\ 1\leq\mu\leq n)$ is a polynomial having all its zeros in $|z|<k,\ k\leq 1$ then for $|z|=1$

$$\left|p^{p}(z)\right| \geq \frac{n(n-1)........(n-t+1)}{\left(1+k^{\mu}\right)^{tt}}\times\left[\max_{|z|=1}|p(z)|+\left\{\left(\left(1+k^{\mu}\right)^t-1\right)k^{-n}\min_{|z|=k}|p(z)|\right\}\right].$$

If we take t=1 in Corollary 3, we get the following result, which improves upon the bound of inequality (1.11) due to Dewan and Upadhye [12].

Corollary4. If $p(z)=a_n z^n+\sum_{j=\mu}^{n} a_{n-j}z^{n-j},\ 1\leq\mu\leq n)$ is a polynomial having all its zeros in $|z|<k,\ k\leq 1$ then for $|z|=1$

$$\left|p'(z)\right| \geq \frac{n}{\left(1+k^{\mu}\right)}\left\{\max_{|z|=1}|p(z)|+\frac{1}{k^{n-\mu}}\min_{|z|=1}|p(z)|\right\}.$$

Remark5. If we take $\mu=1$ in corollary 4, it reduces to an inequality, which sharpen upon the bound of inequality (1.10) due to Govil [11].

1. Lemmas

We need the following lemmas for the proof the theorem.

Lemma 1. If all the zeros of a polynomial of degree n lie in a circular region C and *w* is any zero of $D_{\alpha}p(z)$, then at most one of the points *w* and α may lie outside C.

The above Lemma is due to Laguerre (for reference see [6] , p. 52).

Lemma 2. If . If $p(z)=a_n z^n+\sum_{j=\mu}^{n} a_{n-j}z^{n-j},\ 1\leq\mu\leq n$, is a polynomial having all its zeros in $|z|<k,\ k\leq 1$, and for all $\alpha_1,\alpha_2,\ldots,\alpha_t,\ 1\leq t\leq n$ with $|\alpha_1|\geq k^{\mu}$, $|\alpha_2|\geq k^{\mu}$,......, $|\alpha_t|\geq k^{\mu}$, $1\leq t<n$, and $|z|=1$ then we have

$$\left|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}p(z)\right| \geq \frac{n(n-1)........(n-t+1)}{\left(1+k^{\mu}\right)^{tt}}\left[\left\{\left(\left|\alpha_1\right|-k^{\mu}\right)......\left(\left|\alpha_t\right|-k^{\mu}\right)\right\}\max_{|z|=1}|p(z)|\right]. \quad (2.1)$$

Proof of Lemma 2. If $|\alpha_j| = k^{\mu}$ for at least one *j*; $1 \le j \le t$, then inequality () is trivial. Therefore, we assume that $|\alpha_j| > k^{\mu}$ for *j*; $1 \le j \le t$. In the rest, we proceed by mathematical induction. The result is true for t=1, by inequality (), that means if $|\alpha_1| > k^{\mu}$, then

$$\left|D_{\alpha_1} p(z)\right| \ge n\left(\frac{|\alpha_1| - k^{\mu}}{1 + k^{\mu}}\right)\max_{|z|=1}|p(z)|. \tag{2.2}$$

Now for t=2, since

$$D_{\alpha_1} p(z) = \alpha_1 n a_n z^{n-1} + \mu a_{n-\mu} z^{n-\mu} + \{(\mu+1)a_{n-\mu-1} + \alpha_1(n-\mu)a_{n-\mu}\}z^{n-\mu-1} + \ldots\ldots.\{2\alpha_1 a_2 + (n-1)a_1\}z + \alpha_1 a_1 + n a_0$$

and $|\alpha_1| > k^{\mu}$, $D_{\alpha_1} p(z)$ is a polynomial of degree (n-1). Since all the zeros of p(z) lie in $|z| < k$, $k \le 1$, therefore by Lemma1, all the zeros of $D_{\alpha_1} p(z)$ lie in $|z| < k$, $k \le 1$.

applying Inequality (2.2) to $D_{\alpha_1} p(z)$, a polynomial of degree (n-1), and $|\alpha_2| \ge k^{\mu}$, we conclude that

$$\left|D_{\alpha_2} D_{\alpha_1} p(z)\right| \ge (n-1)\left(\frac{|\alpha_2| - k^{\mu}}{1 + k^{\mu}}\right)\max_{|z|=1}\left|D_{\alpha_1} p(z)\right|.$$

Substituting the term $D_{\alpha_1} p(z)$ from (2.2) in this inequality, we obtain

$$\left|D_{\alpha_2} D_{\alpha_1} p(z)\right| \ge n(n-1)\frac{\left(\left|\alpha_1 - k^{\mu}\right|\right)\left(|\alpha_2| - k^{\mu}\right)}{\left(1 + k^{\mu}\right)^2}\max_{|z|=1}|p(z)|. \tag{2.3}$$

This implies that result is true for t=2. Now we assume that the result is true for t=s<n : it means that for $|z| = 1$ we have

$$\left|D_{\alpha_{st}} ..D_{\alpha_2} D_{\alpha_1} p(z)\right| \ge \frac{n(n-1).......(n-s+1)}{\left(1+k^{\mu}\right)^{ts}}\left[\left\{\left(|\alpha_1| - k^{\mu}\right).....\left(|\alpha_{st}| - k^{\mu}\right)\right\}\max_{|z|=1}|p(z)|\right]. \tag{2.4}$$

Now we shall prove that the result is true for t=s+1<n. According to the above procedure, using Lemma1, the polynomial $D_{\alpha_{st}} ..D_{\alpha_2} D_{\alpha_1} p(z)$ is a polynomial of degree (n-s) for all $\alpha_1, \alpha_2, \ldots, \alpha_{st}$, $1 \le s \le n$, with $|\alpha_1| \ge k^{\mu}$ $|\alpha_2| \ge k^{\mu}$,...... $|\alpha_s| \ge k^{\mu}$,

$1 \le s < n$, and has all zeros in $|z| < k$, $k \le 1$ Therefore, for $|\alpha_{s+1}| \ge k^{\mu}$, applying inequality (2.4) to $D_{\alpha_{st}} ..D_{\alpha_2} D_{\alpha_1} p(z)$, we have

$$\left|D_{\alpha_{s+1}}\left\{D_{\alpha_s} .....D_{\alpha_2} D_{\alpha_1} p(z)\right\}\right| \ge \frac{(n-s)\left(\left|\alpha_{s+1} - k^{\mu}\right|\right)}{1 + k^{\mu}}\max_{|z|=1}\left|D_{\alpha_s} .....D_{\alpha_2} D_{\alpha_1} p(z)\right| \tag{2.5}$$

On combining the inequalities (2.4) and (2.5), we get

$$\left|D_{\alpha_{s+1}} D_{\alpha_s} ......D_{\alpha_2} D_{\alpha_1} p(z)\right| \ge \frac{n(n-1)\ldots(n-s)\left(\left|\alpha_1 - k^{\mu}\right|\right)\ldots\left(\left|\alpha_{s+1} - k^{\mu}\right|\right)}{\left(1 + k^{\mu}\right)^{s+1}} \times \max_{|z|=1}\left|D_{\alpha_s} D_{\alpha_s} ......D_{\alpha_2} D_{\alpha_1} p(z)\right|.tr \tag{2.6}$$

This implies that the result is true for $t = s + 1$. The proof is complete.

3. Proof of the Theorem: Let $m = \min_{|z|=k} |p(z)|$. If p(z) has a zero on $|z| = k$, then m=0 and the result follows from Lemma2. Therefore, we suppose that all the zeros of p(z) lie in $|z| < k$, so that m>0. Now $m \le |p(z)|$ for $|z| = k$, therefore if λ is any real or complex number such that $|\lambda| < 1$, then $\left|\lambda m \left(\frac{z}{k}\right)^n\right| < |p(z)|$ for $|z| = k$.

Since all the zeros of p(z) lie in $|z| < k$, by Rouche's Theorem, we deduce that all the zeros of the polynomial

$G(z) = p(z) - \lambda m \left(\frac{z}{k}\right)^n$ lie in $|z| < k$. Applying Lemma 2 for the polynomial G(z) of degree n which has all zeros in $|z| < k$, and for all $\alpha_1, \alpha_2, \ldots, \alpha_t,\ 1 \le t \le n$ with $|\alpha_1| \ge k^\mu$, $|\alpha_2| \ge k^\mu$,......, $|\alpha_t| \ge k^\mu$, $1 \le t < n$,

$$\left|D_{\alpha_t} \ldots D_{\alpha_2} D_{\alpha_1} G(z)\right| \ge \frac{n(n-1)\ldots\ldots(n-t+1)}{\left(1+k^\mu\right)^{tt}} \left[\left\{\left(|\alpha_1| - k^\mu\right)\ldots\ldots\left(|\alpha_t| - k^\mu\right)\right\} \max_{|z|=1} |G(z)|\right].$$

For $|z| = 1$.

Equivalently

$$\left|D_{\alpha_t} \ldots D_{\alpha_2} D_{\alpha_1} p(z) - \lambda \frac{m}{k^n} \left\{n(n-1)\ldots\ldots(n-t+1)\alpha_1\alpha_2 \ldots\ldots \alpha_t\right\} z^{n-t}\right|$$

$$\ge \frac{n(n-1)\ldots\ldots(n-t+1)}{\left(1+k^\mu\right)^{tt}} \left[\left\{\left(|\alpha_1| - k^\mu\right)\ldots\ldots\left(|\alpha_t| - k^\mu\right)\right\} \max_{|z|=1} \left|p(z) - \lambda m \left(\frac{z}{k}\right)^n\right|\right]. \quad (3.1)$$

By Lemma 1, the polynomial $T(z) = D_{\alpha_t} \ldots\ldots D_{\alpha_2} D_{\alpha_1} G(z$ has all its zeros in $|z| \le k$.

That is to say

$T(z) = D_{\alpha_t} \ldots\ldots D_{\alpha_2} D_{\alpha_1} G(z \ne 0$ for $|z| > k$.

Now substituting G(z) in T(z) above, we conclude that for every λ, with $|\lambda| < 1$ and $|z| > k.$,

$$T(z) = D_{\alpha_t} \ldots\ldots D_{\alpha_2} D_{\alpha_1} p(z) - \lambda \frac{m}{k^n} \left\{n(n-1)\ldots\ldots(n-t+1)\alpha_1\alpha_2 \ldots \alpha_t\right\} z^{n-t} \ne 0. \quad (3.2)$$

Thus for $|z| > k.$,

$$\left|D_{\alpha_t} \ldots\ldots D_{\alpha_2} D_{\alpha_1} p(z)\right| \ge \lambda \frac{m}{k^n} \left\{n(n-1)\ldots\ldots(n-t+1)\alpha_1\alpha_2 \ldots \alpha_t\right\} z^{n-t} .$$

If the above inequality is not true, then there is a point z=z$_0$ with $|z_0| > k.$ such that

$$\left|D_{\alpha_t} \ldots\ldots D_{\alpha_2} D_{\alpha_1} p(z)\right| < \lambda \frac{m}{k^n} \left\{n(n-1)\ldots\ldots(n-t+1)\alpha_1\alpha_2 \ldots \alpha_t\right\} z^{n-t} .$$

Now if we take

$$\lambda = \frac{\left|D_{\alpha_t} \ldots\ldots D_{\alpha_2} D_{\alpha_1} p(z_0)\right|}{\frac{m}{k^n} \left\{n(n-1)\ldots\ldots(n-t+1)\alpha_1\alpha_2 \ldots \alpha_t\right\} z_0^{\,n-t}}. \quad ,$$

Then $|\lambda| < 1$ and with choice of λ, we have $T(z_0) = 0$ for $|z_0| > k$. from (3.2). But this contradict the fact that $T(z) \neq 0$ for $|z| > k$.. Hence for $|z| > k$., we have

$$\left|D_{\alpha_t} \ldots\ldots D_{\alpha_2} D_{\alpha_1} p(z)\right| \geq \lambda \frac{m}{k^n} \{n(n-1)\ldots\ldots(n-t+1)\alpha_1\alpha_2\ldots\alpha_t\} z^{n-t} \tag{3.3}$$

Taking a suitable choice for the argument of λ, in inequality (3.1), we get

$$\left|D_{\alpha_t} \ldots D_{\alpha_2} D_{\alpha_1} p(z)\right| - |\lambda| \frac{m}{k^n} \{n(n-1)\ldots\ldots(n-t+1)|\alpha_1||\alpha_2|\ldots\ldots|\alpha_t|\}|z|^{n-t}$$

$$\geq \frac{n(n-1)\ldots\ldots(n-t+1)}{(1+k^\mu)^t}\left[\{(|\alpha_1| - k^\mu)\ldots\ldots(|\alpha_t| - k^\mu)\}\left(|p(z)| - |\lambda| \frac{m}{k^n}|z|^n\right)\right]. \text{ for } |z| = 1.$$

Thus equivalently for $|z| = 1$,

$$\left|D_{\alpha_t} \ldots D_{\alpha_2} D_{\alpha_1} p(z)\right| \geq \frac{n(n-1)\ldots\ldots\ldots(n-t+1)}{(1+k^\mu)^{tt}}$$

$$\times\left[\{(|\alpha_1| - k^\mu)\ldots\ldots(|\alpha_t| - k^\mu)\}\max_{|z|=1}|p(z)| + |\lambda|\left\{\begin{array}{l}(1+k^\mu)^t(|\alpha_1||\alpha_2|\ldots\ldots|\alpha_t|) \\ -\{(|\alpha \quad| - k^\mu)\ldots(|\alpha_t| - k^\mu)\}\frac{m}{k^n}\end{array}\right\}\right].$$

Finally making $|\lambda| \to 1$, the theorem follows.

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