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Research Article

Two new modular relations for the Göllnitz-Gordon functions

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Abstract In this note, we derive two new modular relations for the Göllnitz-Gordon functions. Our proofs only rely on the Jacobi triple product identity.

Keywords modular relation, Göllnitz-Gordon relation, Jacobi triple product identity.

AMS Subject Classification:11B65, 11F11.

Introduction

Throughout this paper, we let $|q|<1$ and for nonnegative integer n, we use the standard notation

$$(a;q)_n=\prod_{i=0}^{n-1}(1-aq^i),\qquad (a;q)_\infty=\prod_{i=0}^{\infty}(1-aq^i)$$

and

$$(a_1,a_2,\ldots,a_n;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty .$$

The Göllnitz-Gordon functions are defined by

$$S(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2}=\frac{1}{(q,q^4,q^7;q^8)_\infty} \tag{1.1}$$

and

$$T(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2+2n}=\frac{1}{(q^3,q^4,q^5;q^8)_\infty}. \tag{1.2}$$

S(q) and *T(q)* are known as the Göllnitz-Gordon functions, see [1, 2]. Huang [3] and Chen and Huang [4] derived 21 modular relations involving only *S(q)* and *T(q)*. Baruah, Bora and Saikia [5] found new proofs of modular relations for *S(q)* and *T(q)* established by Chen and Huang [3].

The well-known Jacobi triple product identity [6] is

$$\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\binom{n}{2}}=(z,q/z,q;q)_\infty . \tag{1.3}$$

Euler's Pentagonal number theorem is given by

$$f(-q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty ,$$

which is a special case of (1.3). For convenience, denote $f(-q^n)$ by f_n for positive n.

In this note, we derive two new modular relations for the Göllnitz-Gordon functions, which can be stated as follows.

Theorem 1.1 *Let S(q) and T(q) be defined by* (1.1) *and* (1.2), *respectively. We have*

$$S(q^3)S(q)-q^2T(q^3)T(q)=\frac{f_2^2f_6f_8f_{12}}{f_1f_3f_4^2f_{24}}, \tag{1.4}$$

$$S(q^3)T(q)+qS(q)T(q^3)=\frac{f_2f_4f_6^2f_{24}}{f_1f_3f_8f_{12}^2}. \tag{1.5}$$

Proof of Theorem 1.1

In this section, we give an elementary proof of Theorem 1.1. Our proof only relies on the Jacobi triple product identity (1.3). We first establish two lemmas.

Lemma 2.1 *We have*

$$f_1 = f_4 S\left(-q^2\right) - q f_4 T\left(-q^2\right), \tag{2.1}$$

$$\frac{1}{f_1} = \frac{f_4^2}{f_2^3} S\left(-q^2\right) + q\frac{f_4^2}{f_2^3} T\left(-q^2\right). \tag{2.2}$$

Proof. We have

$$f_1 = (q;q)_\infty = \frac{f_2}{f_4}\left(q,q^3,q^4;q^4\right)_\infty = \frac{f_2}{f_4}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2-n} \quad \text{by} \tag{1.3}$$

$$= \frac{f_2}{f_4}\sum_{n=-\infty}^{\infty}(-1)^{2n} q^{2(2n)^2-2n} + \frac{f_2}{f_4}\sum_{n=-\infty}^{\infty}(-1)^{2n+1} q^{2(2n+1)^2-2n-1}$$

$$= \frac{f_2}{f_4}\sum_{n=-\infty}^{\infty} q^{8n^2-2n} - q\frac{f_2}{f_4}\sum_{n=-\infty}^{\infty} q^{8n^2+6n},$$

which yields (2.1) by (1.3). For (2.1), replacing q by $-q$, we see that

$$\frac{f_2^3}{f_1 f_4} = f_4 S\left(-q^2\right) + q f_4 T\left(-q^2\right),$$

which is nothing but (2.2).

Lemma 2.2 *We have*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{2.3}$$

Proof. We have

$$\frac{f_3}{f_1} = \frac{1}{\left(q,q^2;q^3\right)_\infty} = \frac{\left(-q,-q^5,q^6;q^6\right)_\infty}{f_2\left(q^2,q^{10};q^{12}\right)_\infty} = \frac{f_4 f_6}{f_2^2 f_{12}}\sum_{n=-\infty}^{\infty} q^{3n^2-2n} \quad \text{by} \tag{1.3}$$

$$= \frac{f_4 f_6}{f_2^2 f_{12}}\left(\sum_{n=-\infty}^{\infty} q^{3(2n)^2-4n} + \sum_{n=-\infty}^{\infty} q^{3(2n+1)^2-2(2n+1)}\right)$$

$$= \frac{f_4 f_6}{f_2^2 f_{12}}\sum_{n=-\infty}^{\infty} q^{12n^2-4n} + q\frac{f_4 f_6}{f_2^2 f_{12}}\sum_{n=-\infty}^{\infty} q^{12n^2+8n}$$

which yields (2.3) by (1.3).

We now turn to prove Theorem 1.1.

*Proof of Theorem 1.1.*It follows from (2.1), (2.2) and (2.3) that

$$\frac{f_3}{f_1} = \frac{f_4^2 f_{12}}{f_2^3}\left(S\left(-q^6\right) - q^3 T\left(-q^6\right)\right)\left(S\left(-q^2\right) + qT\left(-q^2\right)\right)$$

$$= \frac{f_4^2 f_{12}}{f_2^3}\left(S\left(-q^6\right)S\left(-q^2\right) - q^4 T\left(-q^6\right)T\left(-q^2\right) + q\left(S\left(-q^6\right)T\left(-q^2\right) - q^2 S\left(-q^2\right)T\left(-q^6\right)\right)\right)$$

$$= \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{2.4}$$

Equating even and odd parts of both sides of (2.4), we have

$$S\left(-q^6\right)S\left(-q^2\right) - q^4 T\left(-q^6\right)T\left(-q^2\right) = \frac{f_2 f_6 f_{16} f_{24}^2}{f_4 f_8 f_{12}^2 f_{48}},$$

$$S\left(-q^6\right)T\left(-q^2\right) - q^2 S\left(-q^2\right)T\left(-q^6\right) = \frac{f_2 f_6 f_8^2 f_{48}}{f_4^2 f_{12} f_{16} f_{24}},$$

which implies (1.4) and (1.5) by replacing q^2 by $-q$. This completes the proof.

In fact, we can use the same method to prove more modular relations for $S(q)$ and $T(q)$. For example, from (2.1) and the following identity

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2 f_6} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{2.5}$$

we can derive the following identities which were established by Chen and Huang [4]

$$S\left(q^3\right)S(q) + q^2 T\left(q^3\right)T(q) = \frac{f_3 f_4}{f_1 f_{12}}, \tag{2.6}$$

$$S\left(q^3\right)T\left(q\right)-qS\left(q\right)T\left(q^3\right)=\frac{f_1 f_{12}}{f_3 f_4}. \tag{2.7}$$

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