Available online www.jsaer.com

Journal of Scientific and Engineering Research, 2015, 2(1):28-35

Research Article

ISSN: 2394-2630
CODEN(USA): JSERBR

Hydromagnetic Triply- Diffusive Convection- A Characterization Theorem

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Abstract The problem of triply diffusive magnetoconvection is considered in the present paper. An attempt is made to establish the relationship between various energies in Veronis' type configurations. The analysis made brings out that for Veronis type configuration, the total kinetic energy associated with a disturbance exceeds the sum of its total magnetic and concentration energies in some particular parameter regime. Further, this result is valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

Keywords Triply Diffusive Convection, Rayliegh Numbers, Chandrasekhar Number, Prandtl Number, Magnetic Prandtl Number, Lewis Numbers.

1. Introduction

Thermohaline convection or more generally double diffusive convection has matured into a subject possessing fundamental departure from its counterpart, namely single diffusive convection, and is of direct relevance in the fields of oceanography, astrophysics, liminology and chemical engineering etc. For a broad and a recent view of the subject one may be referred to Brandt and Fernando (1996) [1]. Two fundamental configurations have been studied in the context of thermohaline instability problem, the first one by Stern (1960) wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing and the second one by Veronis (1965) wherein the gradient is destabilizing and the concentration gradient is stabilizing. The main results derived by Stern and Veronis for their respective configurations are that both allow the occurrence of a stationary pattern of motions or oscillatory motions of growing amplitude provided the destabilizing concentration gradient or the temperature gradient is sufficiently large. However, stationary pattern of motion is the preferred mode of setting in of instability in case of Stern's configuration whereas oscillatory motions of growing amplitude are preferred in Veronis' configuration [2-3]. More complicated double-diffusive phenomenon appears if the destabilizing thermal/concentration gradient is opposed by the effect of magnetic field or rotation. Mohan (2010) investigated the problem of thermohaline convection coupled with cross-diffusions for the Veronis type configuration and derived a semi-circle theorem that prescribed upper limits for the complex growth rate of oscillatory motions of neutral or growing amplitude in such a manner that it naturally culminates in sufficient conditions precluding the non- existence of such motions [4]. Kumar and Singh (2010) investigated the Rayleigh-Taylor instability of a Newtonian viscous fluid overlying Walters' B' viscoelastic fluid through porous medium. Kumar and Mohan (2012) have considered the hydromagnetic instability of the plane interface between two uniform, superposed and streaming Rivlin-Ericksen viscoelastic fluids through porous medium [5-6].

All the above researchers have considered the case of two component systems. However, it has been recognized later on by Griffiths (1979)) that there are many situations wherein more than two components are present [7]. Examples of such multiple diffusive convection fluid systems include the solidification of molten alloys, geothermally heated lakes, magmas and their laboratory models and sea water. Griffith (1979), Pearlstein et al. (1989) and Lopez (1990) have theoretically studied the onset of convection in a horizontal layer, of infinite extension of a triply diffusive fluid (where the density depends on three independently diffusing agencies with different diffusivities).These researchers found that small concentrations of a third component with a smaller diffusivity can have a significant effect upon the nature of diffusive instabilities and oscillatory and direct salt finger modes are simultaneously unstable under a wide range of conditions, when the density gradients due to components with the greatest and smallest diffusivity are of same signs .Some fundamental differences between the double and triply convection are noticed by these researchers diffusive. Among these differences, one is that if the gradients of two of the stratifying agencies are held fixed ,then three critical values of the Rayleigh number of the third agency are sometimes required to specify the linear stability criteria(only one critical number is required in double diffusive convection) [8-9]. Another difference is that the onset of convection may occur via a quassiperiodic bifurcation from the motionless basic state. Terrones (1993) studied the effect of cross-diffusion on the stability criteria in a triply diffusive system [10]. Ryzhkov and Shevtsova (2007) studied the case of multicomonent mixture with application to thermogravitational column [11]. Ryzhkov and Shevtsova (2009) also studied the longwave instability of a multicomponent fluid with Soret effect [12]. Rionero (2013a) studied a triply convective diffusive fluid mixture saturating a porous horizontal layer, heated from below and salted from above and obtained sufficient conditions for inhibiting the onset of convection and guaranteeing the global nonlinear stability of the thermal conduction solution [13]. Rionero (2013b) also investigated the multicomponent diffusive convection in porous layer for the more general case when heated from below and salted by m salts partly from above and partly from below [14] . Zhao, Wang and Zhang (2013) investigated the problem of triply diffusive convection in Maxwell fluid saturated porous layer and obtained the criterion for the onset of stationary and oscillatory convection [15]. Shivkumara and Kumar (2013) investigated the bifurcation analysis of a triply diffusive coupled stress fluid in terms of a simplified model consisting of seven nonlinear ordinary differential equations. Shivkumara and Kumar (2014) have studied the linear and weakly nonlinear triple diffusive convection in a couple stress fluid layer [16-17].

Chandrasekhar (1952) in his investigation of magneto hydrodynamic simple Be'nard convection problem sought unsuccessfully the regime in terms of the parameters of the system alone, in which the total kinetic energy associated with a disturbance exceeds the total magnetic energy associated with it, since these considerations are of decisive significance in deciding the validity of the principle of exchange of stabilities. However, the solution for w $(= cons\tan t(\sin \pi z))$ is not correct mathematically (and Chandrasekhar was aware of it) [18]. Banerjee et. al. until 1985 did not pursue their investigation in this direction and consequently did not see this connection. This gap in the literature on magnetoconvection has been completed by Banerjee et. al. (1988) who presented a simple mathematical proof to establish that Chandrasekhar's conjecture is valid in the regime $Q\,\sigma_1 \leq \pi^2$ and further this result is uniformly applicable for any combination of a dynamically free or rigid boundary when the region outside the liquid are perfectly conducting or insulating. Banerjee et al. (1988) showed that in the parameter regime $\frac{Q\sigma_1}{\pi^2} \leq 1$ the total kinetic energy associated with a disturbance is greater than the total magnetic energy associated with it.

Banerjee et. al. (1989) further extended these energy considerations to a more general problem, namely, magnetohydrodynamic thermohaline convection problem, of Veronis type and established that in the parameter regime $\frac{Q\sigma_1}{\pi^2} + \frac{R_S\sigma}{\tau^2\,\pi^4} \leq 1,$ the total kinetic energy associated with a disturbance exceeds the sum of its total magnetic and thermal energies. A similar characterization theorem in magnetothermohaline convection of the Veronis' type was also established by Banerjee et. al in the subsequent year [19-20]. Mohan et al. (2006) derived a characterization theorem in hydromagnetic double diffusive convection and established that the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies in the parameter regime, $\frac{Q\sigma_1}{\pi^2} + \frac{R_S\sigma}{4\tau^2\pi^4} \leq 1$ [21].

The present analysis extends these energies considerations to another complex problem, namely, triply diffusive magnetoconvection problem (analogous to magnetothermohaline convection of the Veronis type) wherein one destabilizing heat component and two stabilizing concentration components have been considered. We establish here that in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{R_S\sigma}{\frac{27}{4}\tau_1^{\,2}\pi^4}+\frac{R'_S\sigma}{\frac{27}{4}\tau_2^{\,2}\pi^4}\le 1$, the total kinetic energy associated with a disturbance exceeds the sum of its total magnetic and concentration energies. Further, this result is valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

2. Mathematical Formulation and Analysis

A viscous and finitely heat conducting Boussinesq fluid is statically confined between two horizontal boundaries z=0 and z=d of infinite horizontal extension and finite vertical depth which are respectively maintained at uniform temperatures T_0 *and* $T_1 (T_0 > T_1)$ and uniform concentrations S_{10}, S_{20} *and* $S_{11}(< S_{10}), S_{21}(,< S_{20})$ in the presence of uniform vertical magnetic field $\vec{H}$.

Following Griffiths (1979) and Banerjee et al. (1989), the relevant governing equations and boundary conditions for the triply diffusive magnetoconvection in their non-dimensional form are given by:

$$\left(D^2-a^2\right)\left(D^2-a^2-\frac{p}{\sigma}\right)w=R_T\,a^2\theta-R_S a^2\phi_1-R'_S a^2\phi_2-QD\left(D^2-a^2\right)h_z\ , \qquad (1)$$

$$\left(D^2-a^2-p\right)\theta=-\ w\ , \qquad (2)$$

$$\left(D^2-a^2-\frac{p}{\tau_1}\right)\phi_1=-\frac{w}{\tau_1}\ , \qquad (3)$$

$$\left(D^2-a^2-\frac{p}{\tau_2}\right)\phi_2=-\frac{w}{\tau_2}\ , \qquad (4)$$

and

$$\left(\mathrm{D}^2-\mathrm{a}^2-\frac{\mathrm{p}\sigma_1}{\sigma}\right)\mathrm{h}_z=-\mathrm{Dw}\,. \qquad (5)$$

with

$w=0=\theta=\phi_1=\phi_2$ on both the boundaries,

$D^2w=0$ on a tangent stress–free boundary everywhere,

$Dw=0$ on a rigid boundary,

$h_z=0$ on both the boundaries if the regions outside the fluid are perfectly conducting,

$$\left.\begin{aligned} Dh_z&=-ah_z\ at\ z=1\\ Dh_z&=ah_z\ at\ z=0\end{aligned}\right\}$$ if the regions outside the fluid are insulating.

(6)

In the above equations (1)–(6), z is real independent variable such that $0 \le z \le 1$, $\mathrm{D}=\frac{\mathrm{d}}{\mathrm{dz}}$ is differentiation w.r.t z , w is the vertical velocity, θ is the temperature, $\phi_1 and \phi_2$ are two concentrations, h_z is the vertical magnetic field, a^2 is the square of the wave number, σ is the Prandtl number, σ_1 is the magnetic Prandtl number, $\tau_1 and \tau_2$ are the Lewis numbers for two concentration components respectively, $R_T>0$ is the thermal Rayliegh number, $R_S > 0$ and $R'_S > 0$ are the concentration Rayliegh numbers for the two concentration components respectively, $p = p_r + ip_i$ is complex constant in general.

We now prove the following theorem:

Theorem 1: If $(p, w, \theta, \phi, h_z)$, $p = p_r + ip_i$, $p_r \geq 0$ is a solution of (1) – (5) together with boundary conditions (6) with $R_T > 0$ $R_S > 0$ $R'_S > 0$ and $\frac{Q\sigma_1}{\pi^2} + \frac{R_S\sigma}{\frac{27}{4}\tau_1^2\pi^4} + \frac{R'_S\sigma}{\frac{27}{4}\tau_2^2\pi^4} \leq 1$, then

$$\int_0^1 \left(|Dw|^2 + a^2|w|^2\right)dz > Q\sigma_1\int_0^1\left(|Dh_z|^2 + a^2|h_z|^2\right)dz + R_S\ a^2\sigma\int_0^1|\phi_1|^2 + R'_S a^2\sigma\int_0^1|\phi_2|^2 dz\ .$$

Proof: Multiplying (5) by h_z^* (the complex conjugate of h_z), integrating the resulting equation over the range of z by parts a suitable number of times, and making use of the boundary conditions (6) we get

$$aM + \int_0^1\left(|Dh_z|^2 + a^2|h_z|^2\right)dz + \frac{p\sigma_1}{\sigma}\int_0^1|h_z|^2 dz = -\int_0^1 w\ Dh_z^*\ , \tag{7}$$

where $M = \left\{\left(|h_z|^2\right)_0 + \left(|h_z|^2\right)_1\right\} \geq 0$.

Equating the real part of (7), we get

$$aM + \int_0^1\left(|Dh_z|^2 + a^2|h_z|^2\right)dz + \frac{p_r\sigma_1}{\sigma}\int_0^1|h_z|^2 dz$$

$$= \text{Real part of}\left(-\int_0^1 w\ Dh_z^*\ dz\right)$$

$$\leq \left|\int_0^1 w\ Dh_z^*\ dz\right|$$

$$\leq \int_0^1 |w||Dh_z|dz$$

$$\leq \left\{\int_0^1|w|^2 dz\right\}^{1/2}\left\{\int_0^1|Dh_z|^2 dz\right\}^{1/2}\ . \tag{8}$$

(using Schwartz inequality)

Since $p_r \geq 0$, therefore from (8), we get

$$\int_0^1|Dh_z|^2 dz < \left\{\int_0^1|w|^2 dz\right\}^{1/2}\left\{\int_0^1|Dh_z|^2 dz\right\}^{1/2}$$

or

$$\int_0^1|Dh_z|^2 dz < \int_0^1|w|^2 dz\ . \tag{9}$$

Using (9), it follows from (8) that

$$\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right)dz < \int_0^1 |w|^2 dz. \qquad (10)$$

Since w (0) = 0 = w (1), therefore using Rayleigh-Ritz inequality (1973), we get

$$\int_0^1 |w|^2 dz < \frac{1}{\pi^2}\int_0^1 |Dw|^2 dz \quad . \qquad (11)$$

It follows from (10) and (11) that

$$\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right)dz < \frac{1}{\pi^2}\int_0^1 |Dw|^2 dz$$

$$< \frac{1}{\pi^2}\int_0^1 \left(|Dw|^2 + a^2|w|^2\right)dz$$

or

$$Q\sigma_1\int_0^1 \left(|Dh_z|^2 + a^2|h_z|^2\right)dz + R_S\sigma\, a^2\int_0^1 |\phi_1|^2 dz + R_S'\sigma\, a^2\int_0^1 |\phi_2|^2 dz$$

$$< \frac{Q\sigma_1}{\pi^2}\int_0^1 \left(|Dw|^2 + a^2|w|^2\right)dz + R_S\quad \sigma\, a^2\int_0^1 |\phi_1|^2 dz + R_S'\quad \sigma\, a^2\int_0^1 |\phi_2|^2 dz$$

. (12)

Multiplying (10) by the complex conjugate of (10) and integrating by parts over the vertical range of z for an appropriate number of times and making use of the boundary conditions (6) for ϕ_1 we get

$$\int_0^1 \left(|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2\right)dz + 2p_r\int_0^1 \left(|D\phi_1|^2 + a^2|\phi_1|^2\right)dz$$

$$+ \frac{|p|^2}{\tau_1^2}\int_0^1 |\phi_1|^2 dz = \frac{1}{\tau_1^2}\int_0^1 |w|^2 dz \quad . \qquad (13)$$

Since, $p_r \geq 0$, therefore, from (13), we get

$$\int_0^1 \left(|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2\right)dz < \frac{1}{\tau_1^2}\int_0^1 |w|^2 dz \; . \qquad (14)$$

Since $\phi_1(0) = 0 = \phi_1(1)$, therefore using Rayleigh-Ritz inequality [1973], we get

$$\pi^2\int_0^1 |\phi|^2 dz < \int_0^1 |D\phi|^2 dz$$

and also

$$\pi^4\int_0^1 |\phi_1|^2 dz \leq \int_0^1 |D^2\phi_1|^2 dz \; . \text{ (using Schwartz inequality)} \qquad (15)$$

It follows from (14) and (15) that

$$\left(\pi^2 + a^2\right)^2\int_0^1 |\phi_1|^2 dz < \frac{1}{\tau_1^2}\int_0^1 |w|^2 dz$$

or

$$\frac{(\pi^2+a^2)^2}{a^2}\int_0^1|\phi_1|^2dz<\frac{1}{a^2\tau_1^2}\int_0^1|w|^2dz<\frac{1}{a^2\tau_1^2(\pi^2+a^2)}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz$$

or

$$a^2\int_0^1|\phi_1|^2dz<\frac{1}{\frac{27}{4}\pi^4\tau_1^2}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz\ ,$$

since the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ for $a^2>0$ is $\frac{27\pi^4}{4}$.

or $$R_S a^2\sigma\int_0^1|\phi_1|^2dz<\frac{R_S\ \sigma}{\frac{27}{4}\pi^4\tau_1^2}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz\ . \qquad (16)$$

Following the same procedure, we have from equation (4), that

$$R_S' a^2\sigma\int_0^1|\phi_2|^2dz<\frac{R_S'\ \sigma}{\frac{27}{4}\pi^4\tau_2^2}\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz \qquad (17)$$

Now from (12), (16) and (17), we get

$$Q\sigma_1\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz+R_S\ a^2\sigma\int_0^1|\phi_1|^2dz+R_S{}' a^2\sigma\int_0^1|\phi_2|^2dz$$

$$<\left(\frac{Q\sigma_1}{\pi^2}+\frac{R_S\sigma}{\frac{27}{4}\tau_1^2\pi^4}+\frac{R_S{}'\sigma}{\frac{27}{4}\tau_2^2\pi^4}\right)\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz\ . \qquad (18)$$

Therefore, if $\frac{Q\sigma_1}{\pi^2}+\frac{R_S\sigma}{\frac{27}{4}\tau_1^2\pi^4}+\frac{R_S{}'\sigma}{\frac{27}{4}\tau_2^2\pi^4}\le 1$ then from (18), we get

$$\int_0^1\left(|Dw|^2+a^2|w|^2\right)dz>Q\sigma_1\int_0^1\left(|Dh_z|^2+a^2|h_z|^2\right)dz+R_S\ a^2\sigma\int_0^1|\phi_1|^2+R_S' a^2\sigma\int_0^1|\phi_2|^2dz$$

(19)

and this completes the proof of the theorem.

We noted that the left hand side of (19) represents the total kinetic energy associated with a disturbance while the right hand side represents the sum of its total magnetic and concentration energies, and Theorem 1 may be stated in the following equivalent form:

At the neutral or unstable state in the triply diffusive magnetoconvection problem of the Veronis' type configuration, the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{R_S\sigma}{\frac{27}{4}\tau_1^2\pi^4}+\frac{R_S{}^{/}\sigma}{\frac{27}{4}\tau_2^2\pi^4}\le 1$ and this result is uniformly valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

3. Conclusions

In the present paper, the hydromagnetic triply diffusive convection problem of Veronis' type configuration is considered. The analysis made brings out the following main conclusion:

At the neutral or unstable state in the hydromagnetic triply diffusive convection problem of the Veronis' type configuration, the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and concentration energies in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{R_S\sigma}{\frac{27}{4}\tau_1^2\pi^4}+\frac{R_S{}^{/}\sigma}{\frac{27}{4}\tau_2^2\pi^4}\le 1$, and this result is uniformly valid for any combination of dynamically free or rigid boundaries that are either perfectly conducting or insulating.

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