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Research Article

DISTRIBUTION OF STRESS IN A SEMI INFINITE ELASTIC MEDIUM DUE TO A TORSIONAL DISPLACEMENT OF THE SURFACE

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Abstract This paper deals with investigation the distribution of stress in a semi infinite elastic medium due to a torsional displacement of the surface.

Keywords Elastic medium, torsional displacement

Introduction

Cinelli [1] has studied the dynamic behaviour of circular plates and beams with internal and external damping; Sharma [2] has obtained the navior type solution for the dynamic response problem of simply supported rectangular plate on Winkler type foundation. Reissner and Sagoic [3,4] began by considering a boundary value problem in the mathematical theory of elasticity.

Main Problem

Distribution of stress in a semi infinite elastic medium due to a torsional displacement of the surface.

The distribution of stress in the interior of a semi infinite elastic medium is determined when a load is applied to the surface by means of a rigid disk, the torsional displacement is prescribed immediately under the disk, and it is assumed that part of the boundary which lies beyond the edge of the disk is free from stress.

To determining the components of stress and displacement in the interior of the semi infinite solid $z \geq o$ when a circular area $r = a$ of the surface is forced to rotate through an angle ϕ about an axis which is normal to the under formed surface of the medium. It is assumed that the region of the surface lying out side the circle $r \leq a$ is free from stress in this case only the circumferential component u_θ of the displacement vector is different from zero and that all the stress components vanish except τ_{z_θ} and τ_{r_θ} which are given by the relation

$$\tau_{z_\theta} = \mu \frac{\partial u_\theta}{\partial z} \tag{1.1.1}$$

$$\tau_{r_\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{1.1.2}$$

Where in the absence of internal damping, u_θ satisfied the partial differential equation

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{1}{C^2}\frac{\partial^2 u_\theta}{\partial t^2} \tag{1.1.3}$$

where

$$C^2 = \frac{\mu}{\rho} \tag{1.1.4}$$

the boundary conditions of the problem are

$$u_\theta = f(r,t), \qquad z = o, \qquad r \le a \tag{1.1.5}$$

$$\tau_{z_\theta} = o, \qquad z = o, \qquad r > a \tag{1.1.6}$$

$$u_\theta(r,z,t)|_{r=a} = f_1(z,t) \tag{1.1.7}$$

$$u_\theta(r,z,t)|_{z=0} = f_2(t) \tag{1.1.8}$$

Solution:

To solve the problem we use finite Hankel transform defined as

$$\bar{f}(\beta_i) = \int_o^b r\ f(r)\, \mathrm{J}_{m_o}(b\ \beta_i)dr \tag{1.1.9}$$

Where β_i is a root of the equation

$$\beta_i \mathrm{J}^I_{m_o}(b\ \beta_i) + h\ \mathrm{J}_{m_o}(b\ \beta_i) = 0 \tag{1.1.10}$$

h is a constant coefficient and J_{m_o} is a bessels function of order m_o of first kind and $\mathrm{J}^I_{m_o}$ derivative of J_{m_o} with respect to r.

Inversion property is given by

$$f(r) = \frac{2}{b}\sum_i \frac{\beta_i^2 \bar{f}(\beta_i)\mathrm{J}_{m_o}(\beta_i, r)}{\left[h^2 + \left\{(\beta_i^2) - \frac{{m_o}^2}{b^2}\right\}\right][\mathrm{J}_{m_o}(b\ \beta_i)]^2} \tag{1.1.11}$$

Where the summation extends over all the positive root of the equation (1.1.10) and operational property is

$$\int_o^b r\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{{m_o}^2 f}{r^2}\right]\mathrm{J}_{m_o}(\beta_i, r) = b\ \mathrm{J}_{m_o}(b\ \beta_i)[f^I(b) + h\ f(b)] - \beta_i^2\ \bar{f}(\beta_i)$$

(1.1.12)

Using finite Hankel transform defined by equation (1.1.9) in equation (1.1.3), we get

$$\int_o^a r\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{1}{r^2}u_\theta\right)J_1(\beta_i\, r)dr-\frac{\partial^2 \bar{u}_\theta}{\partial z^2}=\frac{1}{C^2}\frac{\partial^2 \bar{u}_\theta}{\partial t^2} \qquad (1.1.13)$$

On using operational property (1.1.12), we obtain

$$a\,J_1(a\,\beta_i)\left[u_\theta'(a,z,t)+h\,u_\theta(a,z,t)\right]-\beta_i^2\,\bar{u}_\theta(\beta_i)-\frac{\partial^2 \bar{u}_\theta}{\partial z^2}=\frac{1}{C^2}\frac{\partial^2 \bar{u}_\theta}{\partial t^2} \qquad (1.1.14)$$

Using boundary conditions (1.1.6), we get

$$a\,J_1(a\,\beta_i)[f_1^1(z,t)+h\,f_1(z,t)]-\beta_i^2\,\bar{u}_\theta(\beta_i)-\frac{\partial^2 \bar{u}_\theta}{\partial z^2}=\frac{1}{C^2}\frac{\partial^2 \bar{u}_\theta}{\partial t^2} \qquad (1.1.15)$$

Now using Fourier Sine transform in equation (1.1.15), we get

$$a\,J_1(a\,\beta_i)\left[\int_o^\infty f_1'(z,t)\sin\alpha z\,dz+h\int_o^\infty f_1(z,t)\sin\alpha z\,dz\right]-\beta_i^2\,\bar{\bar{u}}_\theta(\beta_i)-$$

$$\int_o^\infty \frac{\partial^2 \bar{u}_\theta}{\partial z^2}\sin\alpha z\,dz=\frac{1}{C^2}\frac{\partial^2 \bar{\bar{u}}_\theta}{\partial t^2} \qquad (1.1.16)$$

$$\int_o^\infty \frac{\partial^2 T}{\partial x^2}\sin\alpha_n x\,dx=\alpha_n(T)_{x=o}-\alpha_n^2\,\bar{T}(\alpha_n) \qquad (1.1.17)$$

On using the operational property given by the equation (1.1.17), we get

$$a\,J_1(a\,\beta_i)\left[\bar{f}_1'(\alpha,t)+h\,\bar{f}_1(\alpha,t)\right]-\beta_i^2\,\bar{\bar{u}}_\theta(\beta_i,\alpha,t)-\propto[\bar{u}_\theta(z)]_{z=o}-$$

$$r^2\,\bar{\bar{u}}_\theta(\beta_i,\alpha,t)=\frac{1}{C^2}\frac{\partial^2 \bar{\bar{u}}_\theta}{\partial t^2} \qquad (1.1.18)$$

or

$$a\,J_1(a\,\beta_i)\left[\bar{f}_1'(\alpha,t)+h\,\bar{f}_1(\alpha,t)\right]-\beta_i^2\,\bar{\bar{u}}_\theta(\beta_i,\alpha,t)-\propto\bar{f}_2(t)-$$

$$\alpha^2\,\bar{\bar{u}}_\theta(\beta_i,\alpha,t)=\frac{1}{C^2}\frac{\partial^2 \bar{\bar{u}}_\theta}{\partial t^2} \qquad (1.1.19)$$

Now using Laplace transform in equation (1.1.18), we get

$$a\,J_1(a\,\beta_i)\left[\bar{\bar{f}}_1'(\alpha,p)+h\,\bar{\bar{f}}_1(\alpha,p)\right]-\beta_i^2\,\bar{\bar{\bar{u}}}_\theta(\beta_i,\alpha,p)-\propto\bar{\bar{f}}_2(p)-$$

$$\alpha^2\,\bar{\bar{\bar{u}}}_\theta(\beta_i,\alpha,p)=\frac{p^2}{C^2}\bar{\bar{\bar{u}}}(\beta_i,\alpha,p) \qquad (1.1.20)$$

or

$$a\,J_1(a\,\beta_i\,)\left[\bar{\bar{f}}_1'(\alpha,p)+h\,\bar{\bar{f}}_1(\alpha,p)\right]-\ \alpha\bar{\bar{f}}_2(p)=\left(\beta_i^2+\alpha^2+\frac{p^2}{C^2}\right)\bar{\bar{\bar{u}}}_\theta\,(\beta_i,\alpha,p)$$

(1.1.21)

or

$$\bar{\bar{\bar{u}}}_\theta\,(\beta_i,\alpha,p)=\frac{a\,J_1(a\,\beta_i\,)\left[\bar{\bar{f}}_1^1(\alpha,p)+h\,\bar{\bar{f}}_1(\alpha,p)\right]-\ \alpha\bar{\bar{f}}_2(p)}{\left(\beta_i^2+\alpha^2+\frac{p^2}{C^2}\right)} \qquad (1.1.22)$$

Finally it becomes

$$\bar{\bar{\bar{u}}}_\theta\,(\beta_i,\alpha,p)=a\,J_1(a\,\beta_i\,)\,C^2\left[\frac{\bar{\bar{f}}_1^1(\alpha,p)}{C^2(\alpha^2+\beta_i^2)+p^2}+\frac{h\,\bar{\bar{f}}_1(\alpha,p)}{C^2(\alpha^2+\beta_i^2)+p^2}\right]-$$

$$\alpha C^2\,\frac{\bar{\bar{f}}_2(p)}{C^2(\alpha^2+\beta_i^2)+p^2} \qquad (1.1.23)$$

Let

$$C^2(\alpha^2+\beta_i^2)=A^2 \qquad (1.1.24)$$

so

$$\bar{\bar{\bar{u}}}_\theta\,(\beta_i,\alpha,p)=a\,J_1(a\,\beta_i\,)\,C^2\left[\frac{\bar{\bar{f}}_1'(\alpha,p)}{p^2+A^2}+\frac{h\,\bar{\bar{f}}_1(\alpha,p)}{p^2+A^2}\right]-\alpha C^2\,\frac{\bar{\bar{f}}_2(p)}{p^2+A^2} \qquad (1.1.25)$$

Which may be written as

$$\bar{\bar{\bar{u}}}_\theta\,(\beta_i,\alpha,p)=\frac{a\,C^2}{A}\,J_1(a\,\beta_i\,)\,\left[\bar{\bar{f}}_1'(\alpha,p)\frac{A}{p^2+A^2}+h\,\bar{\bar{f}}_1(\alpha,p)\frac{A}{p^2+A^2}\right]-$$

$$\frac{\alpha C^2}{A}\,\bar{\bar{f}}_2(p)\,\frac{A}{p^2+A^2} \qquad (1.1.26)$$

Now using inverse Laplace transform and convolution theorem in equation (1.1.26), we get

$$\bar{\bar{u}}_\theta\,(\beta_i,\alpha,t)=\frac{a\,C^2}{A}\,J_1(a\,\beta_i\,)\left[\int_o^t\bar{f}_1'(\alpha,s)\sin A\,(t-s)\,ds+\right.$$

$$\left.h\,\int_o^t\bar{f}_1\,(\alpha,s)\sin A\,(t-s)\,ds\right]-\frac{\alpha C^2}{A}\,\int_o^t\bar{f}_2\,(s)\sin A\,(t-s)\,ds \qquad (1.1.27)$$

$$f\ (x) = \int_o^\infty \bar{f}\ (\alpha_n) \sin \alpha_n\ x\ d\alpha_n \qquad (1.1.28)$$

Now using inverse Fourier Sine transform given by the equation (1.1.28) in equation (1.1.27), we get

$$\bar{u}_\theta\ (\beta_i, z, t) = \int_0^\infty \left[\frac{a\ C^2}{A} J_1(a\ \beta_i\ ) \left(\int_o^t \bar{f}_1^1\ (\alpha, s) \sin A\ (t-s) ds \right. \right.$$

$$\left. \left. + h \int_o^t \bar{f}_1\ (\alpha, s) \sin A\ (t-s) ds \right) - \frac{\alpha C^2}{A} \int_o^t \bar{f}_2\ (s) \sin A\ (t-s) ds \right] \sin \alpha z\ d\alpha$$

(1.1.29)

$$f(r) = \frac{2}{b} \sum_i \frac{\beta_i^2\ \bar{f}\ (\beta_i) J_{m_o}(\ \beta_i, r)}{\left[h^2 + \left\{(\beta_i^2) - \frac{{m_o}^2}{b^2}\right\}\right] [J_{m_o}(\ b\ \beta_i)]^2} \qquad (1.1.30)$$

Now using inversion theorem of the Hankel transform given by the equation (1.1.30) in equation (1.1.29), we get

$$u_\theta\ (r, z, t) = \frac{2}{a} \int_o^\infty \sum_i \frac{\beta_i^2\ J_1(\ \beta_i\ r)}{\left[h^2 + \left\{(\beta_i^2) - \frac{1}{a^2}\right\}\right] [J_1(a\ \ \beta_i)]} \int_0^\infty \left[\frac{a\ C^2}{A} J_1(a\ \beta_i\ )\right.$$

$$\left(\int_o^t \bar{f}_1'(\alpha, s) \sin A\ (t-s) ds + h \int_o^t \bar{f}_1\ (\alpha, s) \sin A\ (t-s) ds \right) -$$

$$\left. \frac{\alpha C^2}{A} \int_o^t \bar{f}_2\ (s) \sin A\ (t-s) ds \right] \sin \alpha z\ d\alpha \qquad (1.1.31)$$

Where $\bar{f}_1\ (\alpha, s)$ and $\bar{f}_2\ (\alpha, s)$ be the infinite Fourier Sine transform, of $f_1\ (z, s)$ and $f_2\ (z, s)$.

On substituting the value of $u_\theta\ (r, z, t)$ from equation (1.1.31) in equation (1.1.1) and (1.1.2), we get

$$\tau_{z_\theta} = \mu \left(\frac{2}{a} \int_o^\infty \sum_i \frac{\beta_i^2\ J_1(\ \beta_i\ r)}{\left[h^2 + \left\{(\beta_i^2) - \frac{1}{a^2}\right\}\right] [J_1(a\ \ \beta_i)]} \int_0^\infty \left[\frac{a\ C^2}{A} J_1(a\ \beta_i\ )\right.\right.$$

$$\left(\int_o^t \bar{f}_1'(\alpha, s) \sin A\ (t-s) ds + h \int_o^t \bar{f}_1\ (\alpha, s) \sin A\ (t-s) ds \right) -$$

$$\left.\left. \frac{\alpha C^2}{A} \int_o^t \bar{f}_2\ (s) \sin A\ (t-s) ds \right] \alpha\ \cos \alpha z\ d\alpha \right) \qquad (1.1.32)$$

$$\tau_{r_\theta} = \left(\frac{2}{a}\int_o^\infty \sum_i \frac{\beta_i^3\, J_1^I(\beta_i\, r)}{\left[h^2 + \left\{(\beta_i^2) - \frac{1}{a^2}\right\}\right][J_1(a\ \ \beta_i)]} \int_0^\infty \left[\frac{a\ C^2}{A} J_1(a\, \beta_i\ )\right.\right.$$

$$\left(\int_o^t \bar{f}_1^{'}(\alpha, s) \sin A\,(t-s)\, ds + h \int_o^t \bar{f}_1\,(\alpha, s) \sin A\,(t-s) ds\right) - \frac{\alpha C^2}{A}$$

$$\left.\left.\int_o^t \bar{f}_2\,(s) \sin A\,(t-s)\, ds\right] \sin \alpha z\, d\alpha\right) - \frac{1}{r}\ \left(\frac{2}{a}\int_o^\infty \sum_i \frac{\beta_i^2\, J_1(\,\beta_i\, r)}{\left[h^2 + \left\{(\beta_i^2) - \frac{1}{a^2}\right\}\right][J_1(a\ \ \beta_i)]}\right.$$

$$\int_0^\infty \left[\frac{a\ C^2}{A} J_1(a\, \beta_i\ )\left(\int_o^t \bar{f}_1^{'}(\alpha, s) \sin A\,(t-s) ds + h \int_o^t \bar{f}_1\,(\alpha, s) \sin A\,(t-s) ds\right)\right.$$

$$\left.\left. - \frac{\alpha C^2}{A} \int_o^t \bar{f}_2\,(s) \sin A\,(t-s)\, ds\right] \sin \alpha z\, d\alpha\ \right) \qquad (1.1.33)$$

Where equation (1.1.31) represent the component of displacement and (1.1.32), (1.1.33) represent the component of stresses in the interior of the semi infinite solid.

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